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The Constraints of the Valuation Distribution for Solving a Class of Games by Using a Best Response Algorithm

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Abstract

Infinite games with incomplete information are common in practice. First-price, sealed-bid auctions are a prototypical example. To solve this kind of infinite game, a heuristic approach is to discretize the strategy spaces and enumerate to approximate the equilibrium strategies. However, an approximate algorithm might not be guaranteed to converge. This paper discusses the utilization of a best response algorithm in solving infinite games with incomplete information. We show the constraints of the valuation distributions define the necessary conditions of the convergence of the best response algorithm for several classes of infinite games, including auctions. A salient feature of the necessary convergence conditions lies in that they can be employed to compute the exact Nash equilibria without discretizing the strategy space if the best response algorithm converges.

Keywords: auctions, infinite game, e-commerce, decision support systems, best response, necessary conditions, valuation distribution.

1 Introduction

Game models, including auctions, have been extensively applied in value chain management (Vulcano et al., 2002). A great deal of literature concerns itself with computing equilibria in games (see McKelvey and McLennan (1996); Yang (1999)). Scarf's algorithm was the first algorithm developed for approximating a fixed point, and can be used to compute a Nash equilibrium of a game (Scarf, 1967). This was followed by several other algorithms for solving finite games, such as simplicial subdivisition algorithms for finding a sample equilibrium (Talman and Yang, 1994) and semi-algebraic set algorithms for solving all equilibria (McKelvey and McLennan, 1996). However, the computational complexity of these algorithms, in the worst case, is exponential in the dimension and the number of digits of accuracy (Hirsch et al., 1989; McKelvey and McLennan, 1996). Moreover, these algorithms are not applicable to solving infinite games.

Infinite games include games with infinite possible strategies, such as those which have a continuous strategy space.[1] In this paper, we focus on games with a continuous strategy space. To motivate this class of infinite games, researchers have explored game models such as auctions (Klemperer, 2000; Reeves and Wellman, 2004). A heuristic approach is to discretize the strategy space and then to apply existing algorithms for solving finite games to approximate the equilibrium strategies. However, an approximate algorithm might not result in an optimal solution. In the worst case, a discretization algorithm may not converge.

Best response algorithms have been utilized in providing solutions to auctions and supply chain problems (Reeves

[1] Infinite games also include games with infinite agents/players.

and Wellman, 2004). Reeves and Wellman (2004) propose a piece-wise linear strategy function for the best response algorithm and show how their algorithm solves the supply-chain game, first-price sealed-bid auctions, and other games. In this paper, we discuss a similar best response algorithm for two classes of multi-agent symmetric games and a class of 2-agent asymmetric games. We provide the necessary convergence conditions for these classes of games and show how to compute the exact Nash equilibrium strategies in an immediate manner. This approach does not need to discretize the strategy space.

In the next section, we provide the model. In Section 3, the best response algorithm is provided. In Section 4, we further discuss the convergence conditions for both symmetric and asymmetric games. Section 5 illustrates some examples and shows how the computation can be done analytically. We offer some concluding remarks in Section 6.

2 The Model

We assume that there are N agents, each with incomplete information. That is, each agent knows its own true valuation; however, it knows only a distribution function of other agents' valuations. To single out a random agent, we let x be the true valuation of our agent, which we refer to as Agent 0. Let $n = N - 1$. The other agents are indexed from 1 to n. Let v_i be agent i's valuation. Let Y_k be the $(n - k + 1)$-st order statistic of $\{v_1, ..., v_n\}$. Thus, we have $Y_1 \leq Y_2 \leq ... \leq Y_n$. We assume that Y_j is a random independent observation of a continuous and differentiable distribution F_j and its associated probability density function f_j. We let $C_k, k = 1, 2, ...$, be a constant real number.

We assume that the strategy function of agent i, β_i, is a linear function of the agent's true valuation. That is, agent i has $\beta_i = \alpha_i v_i + \gamma_i$, where α_i and γ_i are two coefficients. When a Nash equilibrium to the game is symmetric, in which agents use the same strategy function, α_i and γ_i are the same across all agents, and we use α and γ instead. When the game is asymmetric, we have two sets of coefficients, $\Gamma = \{\gamma_0, \gamma_1, ..., \gamma_n\}$ and $\Xi = \{\alpha_0, \alpha_1, ..., \alpha_n\}$. We assume that the agent having the highest action wins the game, a rule which is compatible with many auction games.

This linear strategy function is a very strong assumption. However, the assumption is reasonable for some special games, including first price sealed bid (FPSB) auctions. Previous work also adopts this similar assumption to approximate the strategies for FPSB auctions (Turocy, 2001). For the sake of comparison, our work follows the same assumption, but aims at deriving the convergence conditions for best response dynamics in infinite games.

In this work, we discuss two different utility functions. The first one is a linear function, $u_i = \theta_i v_i - (\alpha_i v_i + \gamma_i)$, where v_i is the true valuation of agent i and θ_i is the coefficient of v_i, if agent i is the winner. The second one is more general and is expressed as $u_i(v_i, \Gamma, \Xi, \{Y_i\})$, where the utility is a function of our agent's valuation, the other agents' valuations, our agent's action, and the other agents' actions. Although the strategy function, $\beta_i = \alpha_i v_i + \gamma_i$, is linear, the utility function may be non-linear. While the first utility function is a special case of the one proposed by Reeves and Wellman (2004), the second utility function is more general than the first one.

3 The Best Response Algorithm

The best response analysis for infinite games with incomplete information is pre-play reasoning, like fictitious play in games with complete information. While strategies in fictitious play are determined by the frequency of other agents' strategies in the history, strategies in best response analysis are based on the previous step.

The best response algorithm for solving infinite games with incomplete information works as follows.

1. Initiate start seeds of strategies for all agents;
2. For each agent,
   (a) Fix the current strategies of other agents;
   (b) Find the optimal strategy for our agent;
3. Compare new strategies and previous strategies;
   (a) If new strategies are the same as, or close enough to, previous strategies, then stop;
   (b) Otherwise, use the new optimal strategy for our agent to update the strategies for the other agents, and go to Step 2;

In Step 1, we randomly select a pair of $\{\alpha_i^{\{1\}}, \gamma_i^{\{1\}}\}$ for agent i. Step 2 is the core of this best response algorithm. In this work, we do not discretize the strategy space. In the next two sections, we show that we can find the exact best response strategy analytically without going through discretization and enumeration. Step 3 specifies the algorithm's stopping rule. Whether this step will stop is determined by whether the best response procedure converges, the determination of which is the other main topic of this paper.

We discuss two different models: one symmetric and one asymmetric. In the symmetric model, all agents share the common distribution function and use the same strategy function, (i.e., $\alpha_i = \alpha_j$ and $\gamma_i = \gamma_j$), which enable us to simplify the best response procedure. In the first round, the initial strategies for other agents are $\beta = \alpha^{\{1\}} v_i + \gamma^{\{1\}}$. Our agent will find an optimal strategy, $(\alpha^{\{2\}}, \gamma^{\{2\}})$, given other agents' valuation, and their current strategies. In the second round, because we are assuming symmetric strategies, we update the other agents' strategies to $(\alpha^{\{2\}}, \gamma^{\{2\}})$, and then find a new optimal strategy for our agent. This process continues until $(\alpha^{\{k\}}, \gamma^{\{k\}})$ converges to (α, γ).

If this best response procedure converges, according to the definition of Nash equilibrium, the focal point is a Nash equilibrium. However, this best response procedure may not converge. The assumption of F_i plays an important role in solving incomplete information games. Most results in this area are based on assumptions of a particular distribution, such as the uniform distribution used in some examples Reeves and Wellman (2004). It is reasonable to suspect that the form of distribution could impact the convergence of the best response algorithm. In the next section, we focus on the analysis of the necessary convergence conditions by using this best response algorithm and their relations with the distribution functions.

4 Necessary Conditions for the Convergence

In this section, we first discuss the convergence conditions of the best response algorithm for three different classes of infinite games and then show how the necessary conditions for the convergence constrain the valuation functions. The first class is symmetric games with linear strategy functions and linear utility functions. The second class of games is symmetric games with linear strategy functions and general utility functions. The third class is asymmetric games with linear strategy functions and linear utility functions. The discussions of the first two classes of games are applied to multi-agent games, while the discussion of the third class of games is limited to two-agent games.

We start with the symmetric cases with linear utility functions. In the k-th round of the best response procedure, agent i uses the strategy $\beta^{\{k\}}(v_i) = \alpha^{\{k\}} v_i + \gamma^{\{k\}}$. We assume $u_i = \theta v_i - (\alpha v_i + \gamma)$. The probability that our agent has a higher action $\beta^{\{k+1\}}(x)$ in the $k+1$-th round, which is higher than the k-th round strategies of the rest $N-1$ agents, and wins the game is given by $\Pr\left(\beta^{\{k+1\}}(x) > \beta^{\{k\}}(Y_n)\right)$. Thus, our agent's total surplus can be written

$$\begin{aligned}
\Pi &= \Pr(\beta^{\{k+1\}}(x) > \beta^{\{k\}}(Y_n))[\theta x - (\alpha^{\{k+1\}} x + \gamma^{\{k+1\}})] \\
&= \Pr(\alpha^{\{k+1\}} x + \gamma^{\{k+1\}} > \alpha^{\{k\}} Y_n + \gamma^{\{k\}})[\theta x - (\alpha^{\{k+1\}} x + \gamma^{\{k+1\}})] \\
&= \Pr(Y_n < \frac{\alpha^{\{k+1\}} x + \gamma^{\{k+1\}} - \gamma^{\{k\}}}{\alpha^{\{k\}}})[\theta x - (\alpha^{\{k+1\}} x + \gamma^{\{k+1\}})] \\
&= F^n(\frac{\alpha^{\{k+1\}} x + \gamma^{\{k+1\}} - \gamma^{\{k\}}}{\alpha^{\{k\}}})[\theta x - \alpha^{\{k+1\}} x - \gamma^{\{k+1\}}]. \qquad (1)
\end{aligned}$$

By deriving the first order partial differential of equation (1) with respect to $\alpha^{\{k+1\}}$ and $\gamma^{\{k+1\}}$, respectively, we obtain the following necessary condition for finding an optimal converging strategy using the best response algorithm.

$$F(z) = C_2 \left[\frac{n(\theta - \alpha)}{\alpha} z - \frac{n\gamma}{\alpha} \right]^{\frac{\alpha}{n(\theta-\alpha)}}, \tag{2}$$

where we use the transferring function $z = \frac{\alpha^{\{k+1\}}x + \gamma^{\{k+1\}} - \gamma^{\{k\}}}{\alpha^{\{k\}}}$ or $x = \frac{\alpha^{\{k\}}z - \gamma^{\{k+1\}} + \gamma^{\{k\}}}{\alpha^{\{k+1\}}}$. This leads to the following theorem.

Theorem 4.1 *In a class of symmetric games, in which $\beta(v_i) = \alpha v_i + \gamma$ and $u_i = \theta_i x - (\alpha_i x + \gamma_i)$, a necessary condition that the best response procedure converges to a fixed point is that the valuation distribution function must satisfy equation (2).*

Proof: See Appendix.◇

The common first-price, sealed-bid (FPSB) auction is a game to which Theorem 4.1 can be applied. In fact, $u_i = \theta_i x - (\alpha_i x + \gamma_i)$ is a more general payoff function than is typically studied in the literature. Various strategic solutions to FPSB auctions can be found in a large body of research (Klemperer, 1999; Krishna, 2002; McAfee and McMillan, 1987; Milgrom and Weber, 1982; Milgrom, 1989; Vickrey, 1961). Thus, Theorem 4.1 does not provide any new solutions to the traditional FPSB auction, or its extensions, but points out the limitation of the best response algorithm when applied to continuous versions of FPSB games. To ensure the best response algorithm converges, the valuation distribution function is constrained to equation (2).

In symmetric cases with general utility functions, we adopt a more general utility function, $u_i(v_i, \Gamma, \Xi, \{Y_i\})$, while using the same linear strategy function. As in equation (1), we obtain a new surplus function as follows.

$$\Pi = F^n\left(\frac{\alpha^{\{k+1\}}x + \gamma^{\{k+1\}} - \gamma^{\{k\}}}{\alpha^{\{k\}}}\right) u(x, \Gamma, \Xi, \{Y_i\}). \tag{3}$$

Similar to Theorem 4.1, in order for the best response algorithm to converge, the necessary conditions are given by:

$$F^{(1)}(z) = C_3 e^{\int_{-\infty}^{x} \left[-\frac{\alpha}{nx} \frac{\partial \ln u(x,\Gamma,\Xi,\{Y_i\})}{\partial \alpha} \right] dx}, \tag{4}$$

or

$$F^{(2)}(z) = C_4 e^{\int_{-\infty}^{x} \left[-\frac{\alpha}{n} \frac{\partial \ln u(x,\Gamma,\Xi,\{Y_i\})}{\partial \gamma} \right] dx}, \tag{5}$$

and,

$$\frac{\alpha}{n}\int_{-\infty}^{x}\Psi dx = C_5, \tag{6}$$

where $\Psi = \frac{1}{x}\frac{\partial \ln u(x,\Gamma,\Xi,\{Y_i\})}{\partial \alpha} - \frac{\partial \ln u(x,\Gamma,\Xi,\{Y_i\})}{\partial \gamma}$ and $z = \frac{\alpha^{\{k+1\}}x+\gamma^{\{k+1\}}-\gamma^{\{k\}}}{\alpha^{\{k\}}}$. The Hessian matrix of Π for an optimal best response solution $\{\alpha^*, \gamma^*\}$ is given by

$$H(\alpha^*,\gamma^*) = \begin{bmatrix} \frac{\partial^2\Pi}{\partial(\alpha^{\{k+1\}})^2} & \frac{\partial^2\Pi}{\partial\alpha^{\{k+1\}}\gamma^{\{k+1\}}} \\ \frac{\partial^2\Pi}{\partial\gamma^{\{k+1\}}\alpha^{\{k+1\}}} & \frac{\partial^2\Pi}{\partial(\gamma^{\{k+1\}})^2} \end{bmatrix}_{(\alpha^*,\gamma^*)} . \tag{7}$$

Thus, we have the following theorem.

Theorem 4.2 *In a class of symmetric games, in which $\beta(v_i) = \alpha v_i + \gamma$ and $u(x,\Gamma,\Xi,\{Y_i\})$, the necessary condition that the best response procedure converges to a fixed point requires the valuation function must be constrained by equation (4) or (5), equation (6), and that equation (7) is negative.*

Proof: See Appendix.$\diamond$

Theorem 4.2 applies to symmetric game with more general utility functions than does Theorem 4.1. The valuation distribution functions are constrained by equations (4) or (5). Further, when equation (7) is negative, it guarantees that a converging best response solution maximizes agents' payoffs.

Now we discuss the two-agent asymmetric case with linear utility functions. Suppose that we have two agents, 1 and 2. Agent 1 and Agent 2 have true valuations x and y, which are drawn from two different valuation distribution functions F_1 and F_2 respectively. We continue to assume $u_i = \theta_i x - (\alpha_i x + \gamma_i)$. Thus, the surplus functions of both agents are given by the following:

$$\begin{aligned} \Pi_1 &= \Pr(\beta_1(x) > \beta_2(Y))[\theta_1 x - (\alpha_1^{\{k+1\}}x + \gamma_1^{\{k+1\}})] \\ &= \Pr(\alpha_1^{\{k+1\}}x + \gamma_1^{\{k+1\}} > \alpha_2^{\{k\}}Y + \gamma_2^{\{k\}})[\theta_1 x - (\alpha_1^{\{k+1\}}x + \gamma_1^{\{k+1\}})] \\ &= \Pr(Y < \frac{\alpha_1^{\{k+1\}}x + \gamma_1^{\{k+1\}} - \gamma_2^{\{k\}}}{\alpha_2^{\{k\}}})[\theta_1 x - (\alpha_1^{\{k+1\}}x + \gamma_1^{\{k+1\}})] \\ &= F_2(\frac{\alpha_1^{\{k+1\}}x + \gamma_1^{\{k+1\}} - \gamma_2^{\{k\}}}{\alpha_2^{\{k\}}})[\theta_1 x - \alpha_1^{\{k+1\}}x - \gamma_1^{\{k+1\}}] \end{aligned} \tag{8}$$

and

$$
\begin{aligned}
\Pi_2 &= \Pr(\beta_2(y) > \beta_1(X))[\theta_2 y - (\alpha_2^{\{k\}} y + \gamma_2^{\{k\}})] \\
&= \Pr(\alpha_2^{\{k\}} y + \gamma_2^{\{k\}} > \alpha_1^{\{k+1\}} X + \gamma_1^{\{k+1\}})[\theta_2 y - (\alpha_2^{\{k\}} y + \gamma_2^{\{k\}})] \\
&= \Pr(X < \frac{\alpha_2^{\{k\}} y + \gamma_2^{\{k\}} - \gamma_1^{\{k+1\}}}{\alpha_1^{\{k+1\}}})[\theta_2 y - (\alpha_2^{\{k\}} y + \gamma_2^{\{k\}})] \\
&= F_1(\frac{\alpha_2^{\{k\}} y + \gamma_2^{\{k\}} - \gamma_1^{\{k+1\}}}{\alpha_1^{\{k+1\}}})[\theta_2 y - \alpha_2^{\{k\}} y - \gamma_2^{\{k\}}]. \qquad (9)
\end{aligned}
$$

Similarly, the necessary conditions that the best response algorithm obtains its converging optimal solutions are

$$
F_2(z) = C_6 \left[\frac{(\theta_1 - \alpha_1)}{\alpha_1} z - \frac{[\theta_1(\gamma_1 - \gamma_2) + \alpha_1 \gamma_2]}{\alpha_2 \alpha_1} \right]^{\frac{\alpha_1}{(\theta_1 - \alpha_1)}}, \qquad (10)
$$

and

$$
F_1(z) = C_7 \left[\frac{(\theta_2 - \alpha_2)}{\alpha_2} z - \frac{[\theta_2(\gamma_2 - \gamma_1) + \alpha_2 \gamma_1]}{\alpha_1 \alpha_2} \right]^{\frac{\alpha_2}{(\theta_2 - \alpha_2)}}. \qquad (11)
$$

These relations lay the foundation for the following theorem.

Theorem 4.3 *In a class of asymmetric games with two agents, in which* $\beta(v_i) = \alpha v_i + \gamma$ *and* $u_i = \theta_i x - (\alpha_i x + \gamma_i)$*, the necessary condition that the best response procedure converges to a fixed point requires the valuation function must be constrained to equations (10) and (11).*

Proof: See Appendix.◇

Theorem 4.3 discusses a two-agent asymmetric case. We expect a multi-agent version of this result to be much more complicated. Equations (10) and (11) are similar in structure but have differentiated parameters in the distribution functions.

5 Computation and Examples

The above discussions focus on the conditions necessary for convergence when applying a best response algorithm. These conditions are necessary, but not sufficient, to guarantee convergence. However, if the procedure does converge, we may use the conditions to compute the equilibrium in an immediate manner.

To see how, first, examine the valuation distribution function and the necessary conditions for the best response algorithms. If the valuation distribution functions do not satisfy the necessary conditions, the best response algorithm cannot converge. If the necessary conditions are satisfied, we consider two situations. First, consider the symmetric case with linear utility function. If we do not know whether the best response algorithm converges or not, we may use equation (A-4) to compute the series of $\{\alpha^k\}$ step by step until it satisfies the stopping rule. If the best response algorithm does converge, we may directly use equation (2) to compute the converging parameters. We illustrate the computation process with the following examples in a symmetric case with a linear strategy function and a linear utility function.

Example 5.1 *In this example, we consider a case in which the necessary conditions of best response are satisfied. Let the game be a two-person FPSB auction and $\beta(v_i) = \alpha v_i + \gamma$ and $u_i = x - (\alpha_i x + \gamma_i)$. Let Y_i be a random variable of a commonly distributed uniform function,* $U[5, 20]$.

$$F_{y_i} = \begin{cases} 0 & y_i \leq 5 \\ \frac{y_i - 5}{15} & 5 \leq y_i \leq 20 \\ 1 & y_i \geq 20 \end{cases} \tag{12}$$

Suppose we do not know whether the best response algorithm converges. We first compare F_{y_i} to equation (A-4). We find $1 = \frac{\alpha^{\{2\}}}{1-\alpha^{\{2\}}}$ and obtain $\alpha^{\{2\}} = \frac{1}{2}$. Since in the index part $\left(\frac{\alpha^{\{k+1\}}}{n(\theta-\alpha^{\{k+1\}})}\right)$ of equation (A-4), there is only one unknown parameter $\alpha^{\{k+1\}}$, we can get the convergent value of α^* within one step.

To find the value of γ^*, we have a relationship $\gamma^{\{k+1\}} = \frac{5}{2}\alpha^{\{k\}} + \frac{\gamma^{\{k\}}}{2}$ when matching the prior CDF and the necessary condition required by equation (A-4). To see the procedure, without loss of generality, we suppose a start seed $\{\alpha^{\{1\}} = 1, \gamma^{\{1\}} = -1\}$. The convergence procedure is given as following.

$$\alpha^{\{1\}} = 1 \quad \gamma^{\{1\}} = -1$$

$$\alpha^{\{2\}} = \tfrac{1}{2} \quad \gamma^{\{2\}} = 2$$

$$\alpha^{\{3\}} = \tfrac{1}{2} \quad \gamma^{\{3\}} = \tfrac{9}{4}$$

$$\alpha^{\{4\}} = \tfrac{1}{2} \quad \gamma^{\{4\}} = \tfrac{19}{8}$$

$$\alpha^{\{5\}} = \tfrac{1}{2} \quad \gamma^{\{5\}} = \tfrac{39}{16}$$

$$\alpha^{\{6\}} = \tfrac{1}{2} \quad \gamma^{\{6\}} = \tfrac{79}{32}$$

$$\ldots$$

$$\alpha^{\{\infty\}} = \tfrac{1}{2} \quad \gamma^{\{\infty\}} = \tfrac{5}{2}$$

As shown by the above procedure, the best response algorithm converges in this example. If so, we may compute the value directly by comparing equation (2) to equation (12). To see how, we work on the same example.

Example 5.2 *As before, we immediately get the value for* $\alpha^* = \frac{1}{2}$. *As for the value of* γ^*, *we have* $\gamma^* = \frac{5}{2}\alpha^* + \frac{\gamma^*}{2}$ *from the prior CDF and equation (2). Substitute* $\frac{1}{2}$ *for* α^*, *and solve* γ^*. *We obtain* $\gamma^* = \frac{5}{2}$. *We can also obtain the same value by setting* $F(0) = 0$ *and have the equation* $5 = \frac{\gamma^*}{1-\alpha^*}$.

6 Conclusion

Game theory is a useful tool in analyzing value chains. In this paper, we develop the constraints on valuation distributions that provide the necessary conditions for convergence of the best response algorithm for three different classes of infinite games under incomplete information. We discuss the symmetric case with a linear utility function, the symmetric case with a general utility function, and the asymmetric case with a linear utility function. When the best response procedure converges, we show that we can compute the Nash equilibrium directly from the necessary conditions analytically without going through any simulation or discretization of the strategy space.

The assumption of a linear strategy function is a strong assumption. A relaxation of the assumption of linear strategy function is necessary to enable a wider application of best response algorithms in solving infinite games. A possible direction is to use Taylor expansion to replace a general strategy function. However, the computational complexity would become overwhelming due to the large set of coefficients.

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Appendix

Proof of Theorem 4.1: Given the CDF is continuous and differentiable, the surplus function is also continuous and differentiable. The first order condition that a new combination $\{\alpha^{\{k+1\}}, \gamma^{\{k+1\}}\}$ is optimal requires

$$\frac{\partial \Pi}{\partial \alpha^{\{k+1\}}} = 0, \quad \frac{\partial \Pi}{\partial \gamma^{\{k+1\}}} = 0. \tag{A-1}$$

By deriving the first order partial differential of equation (1) with respect to $\alpha^{\{k+1\}}$ and $\gamma^{\{k+1\}}$ respectively, we obtain the following two equations.

$$\begin{aligned}\frac{\partial \Pi}{\partial \alpha^{\{k+1\}}} &= \frac{nx}{\alpha^{\{k\}}} F^{n-1}\left(\frac{\alpha^{\{k+1\}}x + \gamma^{\{k+1\}} - \gamma^{\{k\}}}{\alpha^{\{k\}}}\right) f\left(\frac{\alpha^{\{k+1\}}x + \gamma^{\{k+1\}} - \gamma^{\{k\}}}{\alpha^{\{k\}}}\right)[\theta x - \alpha^{\{k+1\}}x - \gamma^{\{k+1\}}] \\ &\quad - xF^n\left(\frac{\alpha^{\{k+1\}}x + \gamma^{\{k+1\}} - \gamma^{\{k\}}}{\alpha^{\{k\}}}\right),\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \Pi}{\partial \gamma^{\{k+1\}}} &= \frac{n}{\alpha^{\{k\}}} F^{n-1}\left(\frac{\alpha^{\{k+1\}}x + \gamma^{\{k+1\}} - \gamma^{\{k\}}}{\alpha^{\{k\}}}\right) f\left(\frac{\alpha^{\{k+1\}}x + \gamma^{\{k+1\}} - \gamma^{\{k\}}}{\alpha^{\{k\}}}\right)[\theta x - \alpha^{\{k+1\}}x - \gamma^{\{k+1\}}] \\ &\quad - F^n\left(\frac{\alpha^{\{k+1\}}x + \gamma^{\{k+1\}} - \gamma^{\{k\}}}{\alpha^{\{k\}}}\right).\end{aligned}$$

From the above two equations, we observe that $\frac{\partial \Pi}{\partial \alpha^{\{k+1\}}} = x \frac{\partial \Pi}{\partial \gamma^{\{k+1\}}}$. Thus, we get

$$\begin{aligned}0 &= \frac{n}{\alpha^{\{k\}}} F^{n-1}\left(\frac{\alpha^{\{k+1\}}x + \gamma^{\{k+1\}} - \gamma^{\{k\}}}{\alpha^{\{k\}}}\right) f\left(\frac{\alpha^{\{k+1\}}x + \gamma^{\{k+1\}} - \gamma^{\{k\}}}{\alpha^{\{k\}}}\right)[\theta x - \alpha^{\{k+1\}}x - \gamma^{\{k+1\}}] \\ &\quad - F^n\left(\frac{\alpha^{\{k+1\}}x + \gamma^{\{k+1\}} - \gamma^{\{k\}}}{\alpha^{\{k\}}}\right).\end{aligned}$$

By removing the common item $F^{n-1}(\frac{\alpha^{\{k+1\}}x+\gamma^{\{k+1\}}-\gamma^{\{k\}}}{\alpha^{\{k\}}})$ and moving the second part to the right hand side, we get the following:

$$\frac{1}{n\frac{\theta x-\alpha^{\{k+1\}}x-\gamma^{\{k+1\}}}{\alpha^{\{k\}}}} = \frac{f(\frac{\alpha^{\{k+1\}}x+\gamma^{\{k+1\}}-\gamma^{\{k\}}}{\alpha^{\{k\}}})}{F(\frac{\alpha^{\{k+1\}}x+\gamma^{\{k+1\}}-\gamma^{\{k\}}}{\alpha^{\{k\}}})}. \tag{A-2}$$

To simplify, substitute z for $\frac{\alpha^{\{k+1\}}x+\gamma^{\{k+1\}}-\gamma^{\{k\}}}{\alpha^{\{k\}}}$, which results in $x = \frac{\alpha^{\{k\}}z-\gamma^{\{k+1\}}+\gamma^{\{k\}}}{\alpha^{\{k+1\}}}$. Thus, equation (A-2) can be rewritten as

$$\frac{f(z)}{F(z)} = \frac{1}{\frac{n(\theta-\alpha^{\{k+1\}})}{\alpha^{\{k+1\}}}z - \frac{n[\theta(\gamma^{\{k+1\}}-\gamma^{\{k\}})+\alpha^{\{k+1\}}\gamma^{\{k\}}]}{\alpha^{\{k\}}\alpha^{\{k+1\}}}}.$$

The above equation is equivalent to

$$\frac{d(\ln F(z))}{dz} = \frac{\frac{\alpha^{\{k+1\}}}{n(\theta-\alpha^{\{k+1\}})}}{z}\frac{d\big(\ln(\frac{n(\theta-\alpha^{\{k+1\}})}{\alpha^{\{k+1\}}}z - \frac{n[\theta(\gamma^{\{k+1\}}-\gamma^{\{k\}})+\alpha^{\{k+1\}}\gamma^{\{k\}}]}{\alpha^{\{k\}}\alpha^{\{k+1\}}})\big)}{dz}. \tag{A-3}$$

By integrating equation (A-3), we obtain the following:

$$\ln F(z) = \frac{\alpha^{\{k+1\}}}{n(\theta-\alpha^{\{k+1\}})}\ln\big(\frac{n(\theta-\alpha^{\{k+1\}})}{\alpha^{\{k+1\}}}z - \frac{n[\theta(\gamma^{\{k+1\}}-\gamma^{\{k\}})+\alpha^{\{k+1\}}\gamma^{\{k\}}]}{\alpha^{\{k\}}\alpha^{\{k+1\}}}\big) + C_1,$$

This equation can be further refined into the following:

$$F(z) = C_2\Phi^{\frac{\alpha^{\{k+1\}}}{n(\theta-\alpha^{\{k+1\}})}}, \tag{A-4}$$

where $\Phi = \frac{n(\theta-\alpha^{\{k+1\}})}{\alpha^{\{k+1\}}}z - \frac{n[\theta(\gamma^{\{k+1\}}-\gamma^{\{k\}})+\alpha^{\{k+1\}}\gamma^{\{k\}}]}{\alpha^{\{k\}}\alpha^{\{k+1\}}}$.

We conclude that equation (A-4) is a necessary condition for finding an optimal best response strategy given other agents' strategies. $F(z)$ is the CDF of an agent's valuation distribution function. Thus, the CDF must be restricted to the class of functions specified by equation (A-4).

Since $\{\alpha^{\{k\}}, \gamma^{\{k\}}\}$ is not specified before we obtain the above conclusion, this pair of values can either be a random start seed or be a best response seed obtained in a previous round. If this best response process converges, we will have

$\gamma^{\{k\}} = \gamma^{\{k+1\}}$ and $\alpha^{\{k\}} = \alpha^{\{k+1\}}$ in equation (A-4). As a result, equation (A-4) can be rewritten as the following:

$$F(z) = C_2 \left[\frac{n(\theta - \alpha)}{\alpha} z - \frac{n\gamma}{\alpha} \right]^{\frac{\alpha}{n(\theta-\alpha)}}.$$

This proves Theorem 4.1. $\diamond$

Proof of Theorem 4.2: By taking the partial, first-order differential of this equation with respect to $\alpha^{\{k+1\}}$ and $\gamma^{\{k+1\}}$, we obtain

$$\begin{aligned} \frac{\partial \Pi}{\partial \alpha^{\{k+1\}}} &= \frac{nx}{\alpha^{\{k\}}} F^{n-1}(\frac{\alpha^{\{k+1\}}x + \gamma^{\{k+1\}} - \gamma^{\{k\}}}{\alpha^{\{k\}}}) f(\frac{\alpha^{\{k+1\}}x + \gamma^{\{k+1\}} - \gamma^{\{k\}}}{\alpha^{\{k\}}}) u(x, \Gamma, \Xi, \{Y_i\}) \\ &+ F^n(\frac{\alpha^{\{k+1\}}x + \gamma^{\{k+1\}} - \gamma^{\{k\}}}{\alpha^{\{k\}}}) \frac{\partial u(x, \Gamma, \Xi, \{Y_i\})}{\partial \alpha^{\{k+1\}}}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \Pi}{\partial \gamma^{\{k+1\}}} &= \frac{n}{\alpha^{\{k\}}} F^{n-1}(\frac{\alpha^{\{k+1\}}x + \gamma^{\{k+1\}} - \gamma^{\{k\}}}{\alpha^{\{k\}}}) f(\frac{\alpha^{\{k+1\}}x + \gamma^{\{k+1\}} - \gamma^{\{k\}}}{\alpha^{\{k\}}}) u(x, \Gamma, \Xi, \{Y_i\}) \\ &+ F^n(\frac{\alpha^{\{k+1\}}x + \gamma^{\{k+1\}} - \gamma^{\{k\}}}{\alpha^{\{k\}}}) \frac{\partial u(x, \Gamma, \Xi, \{Y_i\})}{\partial \gamma^{\{k+1\}}}. \end{aligned}$$

We set these two equations to zero and obtain:

$$-\frac{nx}{\alpha^{\{k\}}} f(\frac{\alpha^{\{k+1\}}x + \gamma^{\{k+1\}} - \gamma^{\{k\}}}{\alpha^{\{k\}}}) u(x, \Gamma, \Xi, \{Y_i\}) = F(\frac{\alpha^{\{k+1\}}x + \gamma^{\{k+1\}} - \gamma^{\{k\}}}{\alpha^{\{k\}}}) \frac{\partial u(x, \Gamma, \Xi, \{Y_i\})}{\partial \alpha^{\{k+1\}}},$$

and

$$-\frac{n}{\alpha^{\{k\}}} f(\frac{\alpha^{\{k+1\}}x + \gamma^{\{k+1\}} - \gamma^{\{k\}}}{\alpha^{\{k\}}}) u(x, \Gamma, \Xi, \{Y_i\}) = F(\frac{\alpha^{\{k+1\}}x + \gamma^{\{k+1\}} - \gamma^{\{k\}}}{\alpha^{\{k\}}}) \frac{\partial u(x, \Gamma, \Xi, \{Y_i\})}{\partial \gamma^{\{k+1\}}}.$$

Further, we have:

$$\frac{f(\frac{\alpha^{\{k+1\}}x + \gamma^{\{k+1\}} - \gamma^{\{k\}}}{\alpha^{\{k\}}})}{F(\frac{\alpha^{\{k+1\}}x + \gamma^{\{k+1\}} - \gamma^{\{k\}}}{\alpha^{\{k\}}})} = -\frac{\alpha^{\{k\}}}{nx} \frac{1}{u(x, \Gamma, \Xi, \{Y_i\})} \frac{\partial u(x, \Gamma, \Xi, \{Y_i\})}{\partial \alpha^{\{k+1\}}},$$

and

$$\frac{f(\frac{\alpha^{\{k+1\}}x+\gamma^{\{k+1\}}-\gamma^{\{k\}}}{\alpha^{\{k\}}})}{F(\frac{\alpha^{\{k+1\}}x+\gamma^{\{k+1\}}-\gamma^{\{k\}}}{\alpha^{\{k\}}})} = -\frac{\alpha^{\{k\}}}{n}\frac{1}{u(x,\Gamma,\Xi,\{Y_i\})}\frac{\partial u(x,\Gamma,\Xi,\{Y_i\})}{\partial\gamma^{\{k+1\}}}.$$

Let $z = \frac{\alpha^{\{k+1\}}x+\gamma^{\{k+1\}}-\gamma^{\{k\}}}{\alpha^{\{k\}}}$. The above two equations are equivalent to

$$d\ln F^{\{k\}}(z)/dz = -\frac{\alpha^{\{k\}}}{nx}\frac{\partial \ln u(x,\Gamma,\Xi,\{Y_i\})}{\partial\alpha^{\{k+1\}}},$$

and

$$d\ln F^{\{k+1\}}(z)/dz = -\frac{\alpha^{\{k\}}}{n}\frac{\partial \ln u(x,\Gamma,\Xi,\{Y_i\})}{\partial\gamma^{\{k+1\}}}.$$

Thus, we can derive two necessary conditions that the pair $\{\alpha^{\{k+1\}},\gamma^{\{k+1\}}\}$ is a best response as following:

$$F^{\{k\}}(z) = C_3 e^{\int_{-\infty}^{x}\left[-\frac{\alpha^{\{k\}}}{nx}\frac{\partial \ln u(x,\Gamma,\Xi,\{Y_i\})}{\partial\alpha^{\{k+1\}}}\right]dx}, \tag{A-5}$$

and

$$F^{\{k+1\}}(z) = C_4 e^{\int_{-\infty}^{x}\left[-\frac{\alpha^{\{k\}}}{n}\frac{\partial \ln u(x,\Gamma,\Xi,\{Y_i\})}{\partial\gamma^{\{k+1\}}}\right]dx}. \tag{A-6}$$

As we know, the valuation distribution function is known prior. Thus, equation (A-5) should be equivalent to equation (A-6). As a result, we have

$$C_3 e^{\int_{-\infty}^{z}\left[-\frac{\alpha^{\{k\}}}{nx}\frac{\partial \ln u(x,\Gamma,\Xi,\{Y_i\})}{\partial\alpha^{\{k+1\}}}\right]dx} = C_4 e^{\int_{-\infty}^{z}\left[-\frac{\alpha^{\{k\}}}{n}\frac{\partial \ln u(x,\Gamma,\Xi,\{Y_i\})}{\partial\gamma^{\{k+1\}}}\right]dx}.$$

The above equation can be converted to the following:

$$\frac{\alpha^{\{k\}}}{n}\int_{-\infty}^{x}\Psi^{\{k+1\}}dx = C_5, \tag{A-7}$$

where $\Psi^{\{k+1\}} = \frac{1}{x}\frac{\partial \ln u(x,\Gamma,\Xi,\{Y_i\})}{\partial\alpha^{\{k+1\}}} - \frac{\partial \ln u(x,\Gamma,\Xi,\{Y_i\})}{\partial\gamma^{\{k+1\}}}$.

To make sure that we obtain the maximum surplus, we should have a negative Hessian matrix. Suppose we obtain an optimal best response $\{\alpha^*, \gamma^*\}$, the Hessian matrix of Π at point $\{\alpha^*, \gamma^*\}$ is given by

$$H(\alpha^*, \gamma^*) \quad = \quad \left[\begin{array}{cc} \frac{\partial^2 \Pi}{\partial (\alpha^{\{k+1\}})^2} & \frac{\partial^2 \Pi}{\partial \alpha^{\{k+1\}} \gamma^{\{k+1\}}} \\ \frac{\partial^2 \Pi}{\partial \gamma^{\{k+1\}} \alpha^{\{k+1\}}} & \frac{\partial^2 \Pi}{\partial (\gamma^{\{k+1\}})^2} \end{array} \right]_{(\alpha^*, \gamma^*)} .$$

If the best response algorithm converges, equations (A-5) and (A-6) become

$$F^{(1)}(z) = C_3 e^{\int_{-\infty}^{x} \left[-\frac{\alpha}{nx} \frac{\partial \ln u(x,\Gamma,\Xi,\{Y_i\})}{\partial \alpha} \right] dx},$$

and

$$F^{(2)}(z) = C_4 e^{\int_{-\infty}^{x} \left[-\frac{\alpha}{n} \frac{\partial \ln u(x,\Gamma,\Xi,\{Y_i\})}{\partial \gamma} \right] dx},$$

And, equation (A-7) becomes

$$\frac{\alpha}{n} \int_{-\infty}^{x} \Psi dx = C_5,$$

where $\Psi = \frac{1}{x} \frac{\partial \ln u(x,\Gamma,\Xi,\{Y_i\})}{\partial \alpha} - \frac{\partial \ln u(x,\Gamma,\Xi,\{Y_i\})}{\partial \gamma}$. Thus, we obtain the theorem. $\diamond$

Proof of Theorem 4.3: From equations (8) and (9) and following the proof of Theorem 4.1, we obtain the necessary conditions as following.

$$F_2(z) = C_6 \Delta_{\{k+1\}}^{\frac{\alpha_1^{\{k+1\}}}{(\theta_1 - \alpha_1^{\{k+1\}})}}, \tag{A-8}$$

and

$$F_1(z) = C_7 \Delta_{\{k\}}^{\frac{\alpha_2^{\{k\}}}{(\theta_2 - \alpha_2^{\{k\}})}}, \tag{A-9}$$

where

$$\Delta_{\{k+1\}} = \frac{(\theta_1 - \alpha_1^{\{k+1\}})}{\alpha_1^{\{k+1\}}} z - \frac{[\theta_1(\gamma_1^{\{k+1\}} - \gamma_2^{\{k\}}) + \alpha_1^{\{k+1\}} \gamma_2^{\{k\}}]}{\alpha_2^{\{k\}} \alpha_1^{\{k+1\}}},$$

and

$$\Delta_{\{k\}} = \frac{(\theta_2 - \alpha_2^{\{k\}})}{\alpha_2^{\{k\}}} z - \frac{[\theta_2(\gamma_2^{\{k\}} - \gamma_1^{\{k+1\}}) + \alpha_2^{\{k\}} \gamma_1^{\{k+1\}}]}{\alpha_1^{\{k+1\}} \alpha_2^{\{k\}}}.$$

If the best response algorithm converges, equations (A-8) and (A-9) become

$$F_2(z) = C_6 \left[\frac{(\theta_1 - \alpha_1)}{\alpha_1} z - \frac{[\theta_1(\gamma_1 - \gamma_2) + \alpha_1 \gamma_2]}{\alpha_2 \alpha_1} \right]^{\frac{\alpha_1}{(\theta_1 - \alpha_1)}} ,$$

and

$$F_1(z) = C_7 \left[\frac{(\theta_2 - \alpha_2)}{\alpha_2} z - \frac{[\theta_2(\gamma_2 - \gamma_1) + \alpha_2 \gamma_1]}{\alpha_1 \alpha_2} \right]^{\frac{\alpha_2}{(\theta_2 - \alpha_2)}} .$$

This proves Theorem 4.3. $\diamond$

References

Hirsch, M., Papadimitriou, C., Vavasis, S., 1989. Exponential lower bounds for finding Brouwer fixed points. Journal of Complexity 5, 379–416.

Klemperer, P., 1999. Auction theory: A guide to the literature. Journal of Economics Surveys 13, 227–286.

Klemperer, P., 2000. The Economic Theory of Auctions. Edward Elgar Publishing Limited.

Krishna, V., 2002. Auction Theory. Elsevier Science, Academic Press.

McAfee, R. P., McMillan, J., 1987. Auctions and bidding. Journal of Economic Literature 25, 699–738.

McKelvey, R., McLennan, A., 1996. Computation of equilibria in finite games. In: Amman, H., Kendrick, D. A., Rust, J. (Eds.), The Handbook of Computational Economics. Vol. I. Elsevier Science B.V., pp. 87–142.

Milgrom, P. R., 1989. Auctions and bidding: A primer. Journal of Economic Perspectives 3, 3–22.

Milgrom, P. R., Weber, R. J., 1982. A theory of auctions and competitive bidding. Econometrica 50, 1089–1122.

Reeves, D. M., Wellman, M. P., 2004. Computing best-response strategies in infinite games of incomplete information. Tech. rep., University of Michigan.

Scarf, H., 1967. The approximation of fixed points of a continuous mapping. SIAM Journal on Applied Mathematics 15, 1328–1343.

Talman, A., Yang, Z., 1994. A simplicial algorithm for computing proper equilibria of finite games. Center discussion paper 9418, Tilburg University.

Turocy, T. L., 2001. Computation and robustness in sealed-bid auctions. Ph.D. thesis, Northwestern University.

Vickrey, W., 1961. Conterspeculation, auctions, and competitive sealed tenders. Journal of Finance 16, 8–37.

Vulcano, G., van Ryzin, G., Maglaras, C., 2002. Optimal dynamic auctions for revenue management. Management Science 48(11), 1388–1407.

Yang, Z., 1999. Computing Equilibria and Fixed Points. Kluwer Academic Publishers.