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Optimal Policy for Production Systems with Two Flexible Resources and TwoProducts

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Optimal Policy for Production Systems with Two Flexible Resources and Two Products

Abstract

Manufacturing companies are facing increasing volatility in demand. As a result, there has been an emerging need for a flexible multi-period manufacturing system that uses multiple resources to produce multiple products with stochastic demands. To manage such multi-product, multi-resource systems, manufacturers need to make two decisions simultaneously: setting a production quantity for each product and allocating the limited resources dynamically among the products. Unfortunately, although the flexibility design and investment have been extensively studied, the literature has been muted on how to make production and allocation decisions optimally from an operational perspective. This paper attempts to fill this literature gap by investigating a multi-period system using multiple flexible resources to produce two products. We identify the structural property of the cost functions, namely ρ - differential monotone. Based on this property, the optimal production and allocation policy can be characterized by switching curves, which divide the state space into eight or nine sub-regions based on the segmentation of decision rules. We analyze different cases in terms of production costs and resource utilization ratios, and show how they affect the optimal production and allocation decisions. Finally, we compare three heuristic policies to the optimal one to display the advantage of resource flexibility and the effectiveness of a heuristic policy. Supplementary materials are available for this article. Go to the publisher's online edition of *IIE Transaction*, datasets, additional tables, detailed proofs, etc.

Keywords: Production/inventory control; manufacturing flexibility; resource flexibility; optimal allocation policy; dynamic programming

1 Introduction

Most manufacturing companies are facing increasing volatility in demand and demand growth in emerging economies due to increased global competition, shorter business cycles and broader product ranges. To survive in this competitive and fast-changing environment, it is critical to enhance flexibility in adjusting productions to manage supply and demand uncertainty. Manufacturing flexibility is widely recognized as an essential component to achieve competitive advantages in the market place (Jain et al., 2013). According to Chauhan and Singh (2014), manufacturing flexibility that can only be achieved with flexible resources, such as machines and workforce, or capacity and inventory (Chod and Rudi, 2005), is termed resource flexibility. Other types of flexibilities, such as process, operation and product flexibility, depend on flexibility of resources (Karuppan and Ganster, 2004). With the arrival of the big data era, production customization is becoming a popular trend, and hence, poses more challenges to manufacturers. Therefore, improving resource flexibility has been regarded as an effective and efficient method to boost a firm’s ability to match supply with demand and escalate the firm’s competitiveness.

One of the industries that rely heavily on a flexible production capacity is car manufacturing (Chod and Rudi, 2005). An auto manufacturer usually operates many different assembly plants and produces many different vehicles (Jordon and Graves, 1995). Assembly capacity is essentially fixed over a substantial time horizon due to rigid labor contracts and capital requirements. However, the demand for individual vehicles fluctuates. To deal with this uncertainty and better match supply with demand, the manufacturer strives for more manufacturing flexibility, that is, the ability to produce more than one vehicle type with the same capacity. Thus, resource flexibility, whereby a production facility can produce multiple products, is a vital design consideration in multiproduct supply chains facing uncertain demand.

As an example, Honda has invested flexible manufacturing capacity in its North American and global production facilities to more quickly and efficiently respond to changes in market demand. This flexible manufacturing system provides a competitive advantage for Honda due to the efficient utilization of global production resources and increased stability of local manufacturing operations and employment. Honda has nine North American auto assembly lines at seven plant sites, producing 16 distinct Honda and Acura models. All of Honda's North American assembly lines produce multiple vehicles on each line. By designing flexibility into each line, Honda can balance consumer demand with production. Additional benefits include workforce stability and efficient use of resources (Honda, 2015).

One increased pressure that managers face is to generating more revenue with fewer workers and resources, as well as ensuring that the company has sufficient staff with flexible skills, especially when the workforce is limited. This encourages managers to strategically analyze their employees’ expertise and apply cross-training, which uses employees to satisfy labor needs beyond their primary skill, thereby creating scheduling and staffing flexibility. With cross-training, firms are better equipped to recover quickly from disruptions and handle transitions gracefully because they do not need a dedicated staff person for each type of job. Instead, they have a well-rounded team of individuals who can use their varied skills for whatever purpose currently is most urgent, by filling in for each other. For example, if a firm receives a big order unexpectedly, it can shift personnel to augment its production team. This allows a manager to better handle varying product demands with more flexible labor resources.

The application of resource flexibility can also be found in reconfigurable manufacturing systems (RMS), which emphasize the ability to change and evolve rapidly to adjust the production capacity and functionality. Due to competition and the wide range of applications of new technology, manufacturers need to produce a variety of generic or custom-made products to meet customer requirements. These products have a wide range including cellular phones, electronic parts, automobiles, mechanical modules, and others. Such manufacturing systems must meet various customer-dependent requirements, producing specific product variants in small quantities to reduce the stock of finished products. When a sudden change occurs in the market, the RMS changes in response, allowing the company to produce products or goods in an efficient manner. Two main characteristics of RMS are modularity and Customization. These attributes enable the design of a system for the producing a part family rather than a single part. A *part family* is defined as all parts (or products) that have similar geometric features and shapes and the same tolerance level, require the same processes, and are within the same cost range. Because of the linkage between part families, the manufacturer needs to plan its capacity for producing multiple products simultaneously. However, finding the right capacity level for all products at the same time is a challenging problem. A manufacturer, therefore, would benefit from efficient and practical algorithms designed for solving capacity planning and allocation problems.

1.1 Our Main Results and Contributions

Although the management of flexible resources has received significant attention in the operations literature (Chod and Rudi, 2005), very few analytical results have been obtained regarding how to optimally make production and resource allocation decisions in a multiple-period setting. This research seeks to provide theoretical results and managerial insights that answer the following important questions to minimize operation-related costs: What is the optimal production quantity for each product? Given limited resource capacities and different resource utilization ratios, which are defined as the amount of resource needed to produce one unit of a particular type of product, how can resources be efficiently allocated between two products? How do cost parameters or resource utilization ratios affect the optimal policy? Are there simple and efficient heuristic policies for the purpose of implementation?

To address the above research questions, we investigate the optimal production and resource allocation policy for manufacturing systems with flexible resources. Our results are not limited to any specific type of resource. Therefore, throughout the paper we employ a broad definition of resource flexibility that reflects many different type of pooling, or grouping together of resources to maximize advantage or minimize risk, such as flexible process, capacity, inventory, product, raw material, labor or service. More specifically, we consider operational decisions for a finite horizon, periodic review manufacturing system that uses multiple flexible resources to produce two finished products. Each resource can be used to produce both products, but has a limited capacity. Resource utilization ratios might be resource and product specific. The two-product case is interesting in itself with a number of practical applications (Li and Tirupati,1995) and is considered a sub-system in a number of papers, for example, Graves and Tomlin (2003), Kulkarni et al. (2004), and Iravani et al. (2005). Furthermore, as Li and Tirupati (1995) mentioned, the two-product case is more tractable and can be considered a building block for developing more general models. It is useful to note that each product could represent a family of items with similar manufacturing characteristics. This type of aggregation is common in hierarchical approaches to capacity and production planning. Thus, a two-product model could reasonably approximate facilities producing several similar products.

We develop an analytic model using dynamic programming and characterize the system's structural properties, which are helpful to investigate the optimal production and allocation policy. The optimal policy has a very complex structure, and depends on cost and resource utilization ratios. To the best of our knowledge, our work is the first to fully characterize the optimal production and allocation policy for systems with multiple flexible resources and two products in multiple periods dynamically.

We demonstrate that optimal production and allocation can be determined by six monotone switching curves and hedging points. We analyze the structure of the optimal policy for two cases in terms of cost parameters. In the first case, each resource is a primary resource for one particular product and it is more cost-effective to produce this product but more expensive to produce the other product. In the second case, one resource is more cost-effective than the other to produce both products. For both cases, the optimal policies have the following important properties:

- The state space of inventory levels can be segmented into eight (for the first case) or nine (for the second case) regions. Depending on which region the initial state is in, the production and allocation decisions vary accordingly (see Theorems 1 and 2 for additional details).
- There is a threshold for each product such that the product will be produced only if its inventory level does not exceed its threshold.
- For both cases, there exist allocation thresholds that determine whether one resource should be used to produce both products or one particular product. The thresholds are state and cost specific, and thus are defined differently for the two cases.

Our results provide managerial insights on how the optimal policy is affected by cost parameters, and how resource allocation depends on the resource utilization ratio of each resource to produce a specific product. Furthermore, we extend our results to a case with two product-dedicated resources and one flexible resource (see Van Mieghem, 1998), as well as a case where multiple flexible resources are used to produce two products. Finally, based on the theoretical results obtained, we develop simple heuristic policies for practice and demonstrate the significance of resource flexibility.

1.2 Literature Review

Manufacturing flexibility is a multi-dimensional concept and there is no general agreement on its definition. Jain et al. (2013) summarize the various levels of manufacturing flexibility classified by Sethi and Sethi (1990) and Koste and Malhotra (1999) and different dimensions, which include machine flexibility, operation flexibility, process flexibility, and product flexibility, among others. For a comprehensive literature review of manufacturing flexibility, we refer readers to Sethi and Sethi (1990), Beach et al. (2000), and Jain et al. (2013).

There is an extensive body of literature on the various forms of resource flexibility. Jordon and Graves (1995) develop principles on the benefits of process flexibility. They show that a sparse flexibility design, which they called the long chain, yields most of the benefits of the total flexibility, and flexibility configurations with the longest and fewest chains for a given number of links that perform the best. They also develop a simple measure for the flexibility in a given product-plant configuration. Graves and Tomlin (2003) present a framework for analyzing the benefits of flexibility in multistage supply chains. They find two phenomena, stage-spanning bottlenecks and floating bottlenecks, which reduce the effectiveness of a flexibility configuration. They identify flexibility guidelines that perform very well for multistage supply chains. These guidelines employ and adapt Jordan and Graves' (1995) single-stage chaining strategy to multistage supply chains. Iravani et al. (2005) present a new perspective on flexibility in manufacturing and service operations by exploring a type of operational flexibility using the concept of 'structural flexibility'. They focus on strategic-level issues of how flexibility can be created by using multipurpose resources such as cross-training labor, flexible machines, or flexible factories. However, these papers present limited analytical results.

Some papers establish theoretical results to explain the effectiveness of flexibility. For example, Chou et al. (2010) provide theoretical results of the performance of the well-known chaining strategy, and identify conditions to guarantee that a sparse process structure can perform nearly as well as a dense full flexibility system. Chou et al. (2011) use the concept of graph expansion to examine how to design a flexible process structure for manufacturing systems. They analyze the worst-case performance of the flexible structure design problem and prove that when demands are bounded by a constant, there exists sparse flexibility with a graph expansion property that achieves sales close to that of full flexibility. Aksin and Karaesmen (2007) identify preferred flexibility structures in service or manufacturing systems. They provide analytical comparison results on flexibility structure and establish certain flexibility design principles for service and manufacturing systems. Simchi-Levi and Wei (2012) also study the structural properties of long chains. They develop a theory that explains the effectiveness of long chain designs for finite size systems. They uncover the supermodularity property of long chains, which shows marginal benefit and shows that the performances of the long chain is characterized by the difference between the performances of two open chains.

Whereas the existing literature on resource flexibility has focused on the determination of a cost-effective flexibility configuration that is likely to meet demand, operational related issues, such as how to allocate flexible resources and how much product to produce by using each resource, are not well studied. In this paper we investigate optimal operational decisions for specific manufacturing systems that use multiple flexible resources to produce two products. Our work is an extension of papers that study the production control of systems that use a single flexible resource to produce multiple products. For periodic review systems, DeCroix and Arreola-Risa (1998) study multiple-product infinite horizon systems where demand is uncertain and the products share a finite resource every period. They characterize the optimal policy for the special case of homogeneous products when all product inventory levels are below the corresponding base-stock levels. Ceryan et al. (2013) consider a firm producing two products with uncertain demands utilizing limited product-dedicated and flexible resources and characterize the structure and sensitivity of the optimal production and pricing decisions. Gong and Chao (2013) study the optimal control policy for capacitated periodic-review inventory systems with remanufacturing, where the serviceable products can be either manufactured from raw materials or remanufactured from returned products, and the system has finite capacities. Many published studies focus on systems in the context of continuous time formulated as a multiclass make-to-stock queue, such as Zheng and Zipkin (1990), Wein (1992), Veatch and Wein (1996), Ha (1997), and Vericourt et al. (2000).

One closely related paper is Chen (2004), who studies stochastic two-item, single-facility, flexible manufacturing systems. He proves that the hedging point policy is generally optimal for systems involving both finite and infinite horizon cases of the problems. Feng et al. (2015) extend Chen (2004) by considering a capacitated multiple-product system where stochastic demand distribution, production rate, unit production cost and periodic expected inventory cost are the same for all the various products. We use production settings similar to Chen (2004) but extend his work by considering two flexible resources, which can be extended to multiple resources that are used to produce two products. Similar to Chen (2004), we use structural properties, called ρ -difference monotone, to characterize the optimal production and allocation policy. Despite the similarity, our work is significantly different with Chen (2004) in the following ways: First, Chen (2004) is limited to a single-machine problem so his results do not address ways to allocate resources and make production plan for flexible manufacturing systems with multiple resources. Our work focused on systems with two flexible resources but it can be extended to multiple resources. Second, as we will see, multi-resource problems, even only for two flexible resources, will cause significant complexity for both analysis and results. When considering two or more flexible resources, the availability of each resource, its productivity and cost-effectiveness to produce a specific product must be considered. The interaction among these issues does not occur in the single-machine systems. Technically, we show that even for two-resource systems, the optimal production and resource allocation policy is very complex: it is characterized by eight or nine sub-regions of the state space, which are determined by a set of switching curves and three hedging points. For multiple-resource problems, we indicate that the same rule as in the two-resource problems could apply. However, increasing resources will cause significant complexity and characterizing the optimal policy becomes impractical. Third, we shed lights on how the productivity and costs of each resource used to produce different products will affect resource allocation and production decisions. We believe this is an important issue in practice, although it does not appear in the study of the single-machine problems, and is ignored in the resource-flexibility literature due to its technical complexity. Fourth, we propose three heuristic policies and numerically test their performance by comparing them with the optimal policy.

Some papers study capacity allocation problems for two-resource, two-product capacitated manufacturing systems. For instance, Kulkarni et al. (2004) study the optimization of network configuration decisions including strategic choices of allocating manufacturing activity and process competence to a set of plants. They show that process plant networks offer significant risk pooling advantages in a wide range of conditions. Bish et al. (2005) investigate capacity allocation for a stylized two-plant, two-product capacitated manufacturing system. They study the performance of two capacity flexibility configurations: nonflexible, where each plant is dedicated to a single product, and fully flexible, where both plants can make both products. They show that the performance of the system depends heavily on the allocation mechanism used to assign products to the available capacity. Unlike Kulkarni et al. (2004) who study single-period, two-stage stochastic optimization problems, and Bish et al. (2005) who study easily-implementable static policies, we investigate optimal allocation and production policies for multiple-resource, two-product systems dynamically in a finite horizon.

More recently, Attia et al. (2012) presents the workforce planning and scheduling problem, with two levers of flexibility at a time, one related to the working time modulation, and the other to the various tasks that can be performed by a given resource. Iravani et al. (2014) study the joint control of production in a flexible process and inventory management via the one-way product substitution of finished goods. They model a dynamically controlled two-product, make-to-stock system with stochastic processing times, and characterize a complex joint optimal production and post-production policy for a special case. Kouvelis and Tian (2014) study a a three-stage sequential decision model regarding investing in flexible capacity, capacity allocation to individual products, and eventual production quantities and pricing in meeting uncertain demand. Jakubovskis (2017) presents a robust optimization model in the context of optimal choices of product-dedicated and flexible capacities in a spatial setting. The paper examines the impact of three critical factors that lead to different capacity utilization and resource flexibility outcomes. Lian et al. (2018) investigate a multi-skilled worker assignment problem in the context of *seru* production systems, in which differences in workers' skill sets and proficiency levels are considered. They develop a meta-heuristic algorithm to solve the problem.

The remainder of this paper is organized as follows. In Section 2, we introduce the problem and develop the analytical model. In Section 3, we study the structural properties of the optimal production and allocation policy. In Section 4, we propose heuristic policies and conduct numerical study to evaluate the heuristics' performance. We extend our discussions in Section 5 and conclude in Section 6. Proofs are included in the Appendix.

2 Model Formulation

We consider a finite horizon, periodic-review production system with two products, which are referred to as product P_1 and product P_2, respectively. Two types of resources, namely R_1 and R_2, are used to produce the two products. In each period, the firm purchases resource R_i from a capacitated supplier with a maximum procurement cost $M_i, i=1,2$. ρ_{ij} is the amount of R_i required to produce one unit of P_j, $i, j=1,2$. Since we use a broad definition of resource, the meaning of ρ_{ij} could be different when the interpretation of the resource is different. For example, the resource could be the amount of raw material/component *i* used to produce product *j*, or it could be the time required for a cross-trained worker *i* to finish task *j*, or ρ_{ij}^{-1} could be the production rate if machine *i* is used to produce product *j*. We assume that products and resources can be real numbers, and fractional production is permitted if producing a whole unit of a product is impossible due to limited resources or is not cost effective.

Define ρ_{11}/ρ_{12} (ρ_{21}/ρ_{22}) as the comparative resource utilization ratio of R_1 (R_2) to produce the two products, and ρ_{11}/ρ_{21} (ρ_{12}/ρ_{22}) as the comparative resource utilization ratio of R_1 and R_2 to produce product 1 (2). It is obvious that an ρ_{11}/ρ_{12} smaller than 1 means we need to assign fewer R_1 to produce one unit P_1 than to produce one unit P_2, and an ρ_{11}/ρ_{21} smaller than 1 means that if only one type of resource is used to produce one unit P_1, then fewer R_1 is needed than R_2.. In practice the resource utilization ratio can be determined by the production complexity of the two products or merely their physical features, such as product size. For example, two machines that have similar functions are used to produce two products. Machine 1 is more efficient so its production time for each product is less than that of machine 2. Product 1 is a high-end product, so its production process is more complex and requires more production time than product 2 when using the same machine. In this case ρ_{ij} is interpreted as the production time using machine *i* to produce product *j*, and we have that $\rho_{1j} < \rho_{2j}$ and $\rho_{i1} > \rho_{i2}$, $i, j=1,2$. Since ρ_{i1}/ρ_{i2}, $i=1,2$, is mainly determined by products' features, it is meaningful to assume that this ratio is identical, regardless of which resource is used. Similarly, since ρ_{1j}/ρ_{2j}, $j=1,2$, is mainly determined by the comparative efficiency of the two resource to produce the same product, it is meaningful to assume that the ratio is identical for the two products. Therefore, throughout the paper we assume that $\rho_{11}/\rho_{12} = \rho_{21}/\rho_{22}$.

It is worth noting that a general ρ_{ij} will lead to technical complexity. In fact, for tractability, most papers in the related literature assume identical ρ_{ij} for any *i* and *j* (i.e., $\rho_{ij}=1$): see for example, Ha (1997), Graves and Tomlin (2003), Chou et al. (2010), and Simchi-Levi and Wei (2012). We relax this assumption by allowing a general ρ_{ij} subject to the constraint that $\rho_{11}/\rho_{12} = \rho_{21}/\rho_{22}$. To the best of our knowledge, this paper is the first to assume a general ρ_{ij} and investigates the impact of the resource utilization ratio on production and resource allocation. Theoretically, this assumption ensures the structural properties that are critical to characterizing the optimal policy can be preserved throughout the time horizon.

The demand for both products, denoted by D_j ($j=1,2$), are stochastic, independently and identically distributed (i.i.d.) from period to period with a mean λ_j. At the beginning of each period, the management decides the production quantities of each product, and the amount of each resource assigned to produce each product, which is subject to the available capacity of this resource. We assume there is no production lead time, which means products will be available in the same period. Unmet demands are backlogged and will be satisfied in the upcoming periods.

The system incurs the following costs: (1) c_{ij}: the production cost for using R_i to produce one unit of P_j ($i, j=1,2$). The interpretation of the production cost could be different depending on the resource type. It may refer to, but is not limited to, one or more of the following costs: raw material, manufacturing, and labor. c_{ij} could be different over *i* and *j*, because, for example, producing with a more efficient machine might be more expensive, or a high-end product might require a more complex and costly manufacturing process. (2) $L_1(x)$ and $L_2(y)$: the one-period inventory holding and backordering cost functions for products 1 and 2, given their inventory levels are *x* and *y*, respectively. We assume $L_1(x)$ and $L_2(y)$ are convex functions, and non-decreasing in the amount of stock on-hand or backorders.

Our objective is to identify the optimal resource allocation and production policy over a time horizon of *T* periods, such that the total expected discounted cost is minimized. Let α ($0<\alpha\le 1$) be the discount factor. Define $f_t(x,y)$ as the minimum expected discounted cost in period *t* given the inventory levels of the two products are *x* and *y*, respectively, at the beginning of period *t*. The problem can be formulated using the following optimality equation for $t=1,\dots,T$:

$$f_t(x,y)=\min_{\substack{\rho_{11}(x'-x)+\rho_{12}(y'-y)\le M_1\\ \rho_{21}(X-x')+\rho_{22}(Y-y')\le M_2\\ X\ge x'\ge x,Y\ge y'\ge y}}\{c_{11}(x'-x)+c_{12}(y'-y)+c_{21}(X-x')+c_{22}(Y-y')$$
$$+L_1(X)+L_2(Y)+\alpha Ef_{t+1}(X-D_1,Y-D_2)\},$$

which can be equivalently written as

$$f_t(x,y)=\min_{\substack{\rho_{11}(x'-x)+\rho_{12}(y'-y)\le M_1\\ x'\ge x,y'\ge y}}\{c_{11}(x'-x)+c_{12}(y'-y)+J_t(x',y')\}, \tag{1}$$

where

$$J_t(x',y')=\min_{\substack{\rho_{21}(X-x')+\rho_{22}(Y-y')\le M_2\\ X\ge x',Y\ge y'}}\{c_{21}(X-x')+c_{22}(Y-y')$$
$$+L_1(X)+L_2(Y)+\alpha Ef_{t+1}(X-D_1,Y-D_2)\}. \tag{2}$$

x', y', X and Y are four decision variables, where x' and y' represent updated inventory levels of the two products after only R_1 is used, while X and Y are inventory levels of the two products, respectively, after R_1 and R_2 are used but before demand is realized. Although we reformulate the problem as a sequential optimization problem, allocation decisions regarding the two resources and productions decisions regarding the two products are made simultaneously. At the end of the time horizon, we assume that $f_{T+1}(x,y)=0$.

Define $\bar{f}_t(x,y)=c_{11}x+c_{12}y+f_t(x,y)$ and $\bar{J}_t(x',y')=c_{21}x'+c_{22}y'+J_t(x',y')$, Eq. (1) and (2) can be written as

$$\bar{f}_t(x,y)=\min_{\substack{\rho_{11}(x'-x)+\rho_{12}(y'-y)\le M_1\\ x'\ge x,y'\ge y}}\{(c_{11}-c_{21})x'+(c_{12}-c_{22})y'+\bar{J}_t(x',y')\},\tag{3}$$

$$\begin{aligned}\bar{J}_t(x',y')=&\min_{\substack{\rho_{21}(X-x')+\rho_{22}(Y-y')\le M_2\\ X\ge x',Y\ge y'}}\{(c_{21}-\alpha c_{11})X+(c_{22}-\alpha c_{12})Y+L_1(X)\\&+L_2(Y)+\alpha E[\bar{f}_{t+1}(X-D_1,Y-D_2)]\}+\alpha c_{11}\lambda_1+\alpha c_{12}\lambda_2.\end{aligned}\tag{4}$$

By defining $\bar{G}_t(x,y)=(c_{11}-c_{21})x+(c_{12}-c_{22})y+\bar{J}_t(x,y)$ and

$$\begin{aligned}\tilde{G}_t(x,y)=&(c_{21}-\alpha c_{11})x+(c_{22}-\alpha c_{12})y+L_1(x)+L_2(y)\\&+\alpha E[\bar{f}_{t+1}(x-D_1,y-D_2)]+\alpha c_{11}\lambda_1+\alpha c_{12}\lambda_2,\end{aligned}$$

we can rewrite Eq. (1) and (2) as

$$\bar{f}_t(x,y)=\min_{\substack{\rho_{11}(x'-x)+\rho_{12}(y'-y)\le M_1\\ x'\ge x,y'\ge y}}\{\bar{G}_t(x',y')\},\tag{5}$$

and

$$\bar{J}_t(x',y')=\min_{\substack{\rho_{21}(X-x')+\rho_{22}(Y-y')\le M_2\\ X\ge x',Y\ge y'}}\{\tilde{G}_t(X,Y)\}.\tag{6}$$

Throughout this paper, we denote the partial derivative of a generic function $v(x,y)$ with respect to the i^{th} variable as $v^i(x,y)$, $i=1,2$. We use $\uparrow$ and $\downarrow$ to denote non-decreasing and non-increasing, respectively. We list all notations in Table 1.

3 Characterization of the Optimal Policy

For a special case with only one resource and two products, Chen (2004) characterizes the optimal resource allocation and production policy as a hedging point policy. A hedging point represents a global minimization point. In the single resource case, three monotone curves interact at the hedging point, dividing the state space into three regions. The optimal production policy is completely determined by the hedging point and the three curves (please refer to Appendix A for a review of this policy). In this section, we will show how the hedging point policy is helpful when studying the optimal policy for this model based on discussions of different cases in terms of production costs.

Table 1: List of Notation.

Notation	Description
T	Time period, $t=1,\ldots,T$
P_j	Product j, $j=1,2$
R_i	Resource i, $i=1,2$
M_i	Capacity of resource i
ρ_{ij}	The amount of R_i used to produce one unit of P_j, $i, j=1,2$
D_j	Demand of product j in a period, a random variable,
λ_j	Mean of D_j
c_{ij}	Unit cost if R_i is used to produce one unit of P_j, $i, j=1,2$
$L_j(\cdot)$	One period inventory holding and backordering cost functions for product j
α	Discount factor, $0<\alpha\le 1$
$q_{ij}^*(x,y)$	The optimal production quantity of P_j by using R_i, $i, j=1,2$, given inventory levels of the two products are x and y, respectively
$f_t(x,y)$	The minimum expected discounted cost in period t given inventory levels of the two products at the beginning of period t are x and y, respectively

3.1 ρ - differential Monotone Properties and Switching Curves

We investigate how the hedging point policy can be applied to characterize the resource allocation and production policy for a general production system with two types of capacitated resources (e.g., raw materials, facilities). It is easy to check that Eq. (5) and (6) have exactly the same format as Eq. (7). Therefore, with the assumption of $\rho_{11}/\rho_{12}=\rho_{21}/\rho_{22}$, we can use the same method in the single-machine model to prove inductively that cost functions can preserve the ρ - differential monotone properties through the whole time horizon. Thus, we have the following results.

Lemma 1. *For* $t=1,\ldots,T$ *,* $\tilde{G}_t(x,y)$ *and* $\bar{G}_t(x,y)$ *are* ρ *- differential monotone, that is,*

(I) $\bar{G}_t^1(x,y)\uparrow x\uparrow y$, $\tilde{G}_t^1(x,y)\uparrow x\uparrow y$; *(II)* $\bar{G}_t^2(x,y)\uparrow x\uparrow y$, $\tilde{G}_t^2(x,y)\uparrow x\uparrow y$;

(III) $\rho_{11}^{-1}\bar{G}_t^1(x,y)-\rho_{12}^{-1}\bar{G}_t^2(x,y)\uparrow x\downarrow y$, $\rho_{21}^{-1}\tilde{G}_t^1(x,y)-\rho_{22}^{-1}\tilde{G}_t^2(x,y)\uparrow x\downarrow y$.

By the same token, there are monotone switching curves for $\bar{G}_t(x,y)$ and $\tilde{G}_t(x,y)$ that could help characterize the optimal policy. Define

$$\bar{S}_1(y)=\sup\{x:\bar{G}_t^1(x,y)\le 0\},$$

$$\bar{S}_2(x)=\sup\{y:\bar{G}_t^2(x,y)\le 0\},$$

$$\bar{S}_0(y)=\sup\{x:\rho_{11}^{-1}\bar{G}_t^1(x,y)-\rho_{12}^{-1}\bar{G}_t^2(x,y)\le 0\},$$

$$\tilde{S}_1(y)=\sup\{x:\tilde{G}_t^1(x,y)\le 0\},$$

$$\tilde{S}_2(x) = \sup\{y : \tilde{G}_t^2(x, y) \le 0\},$$

$$\tilde{S}_0(y) = \sup\{x : \rho_{21}^{-1}\tilde{G}_t^1(x, y) - \rho_{22}^{-1}\tilde{G}_t^2(x, y) \le 0\}.$$

The first three functions and the last three functions represent three switching curves for $\bar{G}_t(x, y)$ and $\tilde{G}_t(x, y)$, respectively, which can be interpreted similarly as $S_1(y)$, $S_2(x)$ and $S_0(y)$ in the single-resource model. We can show that these curves have similar monotone properties as in Chen (2004). Consider $\bar{G}_t(x, y)$; for example, $\bar{S}_1(y)$ is a non-increasing function of *y* and the non-increasing rate does not exceed the changing rate of *y*, $\bar{S}_2(x)$ is a non-increasing function of *x* and the non-increasing rate does not exceed the changing rate of *x*, and $\bar{S}_0(y)$ is a non-decreasing function of *y*. The three curves interact at a hedging point, namely ($\bar{x}^*, \bar{y}^*$). We use ($\tilde{x}^*, \tilde{y}^*$) to represent the hedging point; or the interaction of the three curves of $\tilde{G}_t(x, y)$. By definition, $\bar{G}_t^1(\bar{x}^*, \bar{y}^*) = \bar{G}_t^2(\bar{x}^*, \bar{y}^*) = 0$ and $\tilde{G}_t^1(\tilde{x}^*, \tilde{y}^*) = \tilde{G}_t^2(\tilde{x}^*, \tilde{y}^*) = 0$. Therefore, $\bar{J}_t^1(\bar{x}^*, \bar{y}^*) = c_{21} - c_{11}$ and $\bar{J}_t^2(\bar{x}^*, \bar{y}^*) = c_{22} - c_{12}$.

From Eq. (3) and (4), we can see that the optimal production quantities using R_1 depend on the optimal production quantities using R_2, and vice versa. Unlike the single-resource model, in the two-resource model we must consider the interaction of the production quantities from the two resources, which complicates the optimal policy. Indeed, to the end of this section, we will show that to characterize the optimal policy, we need to use the six switching curves, some auxiliary curves/line segments, and three hedging points in the following subsections. Based on these curves/line segments and hedging points, the state space can be segmented into at least eight regions, each of which represents a set of states that follow the same production and allocation rule (e.g., using R_1 only to produce both products). These curves/line segments and hedging points are also useful to determine the production quantities from each resource analytically.

In addition to the complete characterization of the optimal production and allocation policy, another key feature of our model, which is important in practice but is not well studied in the literature, is the investigation of the impact of product/resource specific costs and resource utilization ratios on the optimal policy. Again, the impact of these costs and ratios can be visualized and identified by the switching curves/line segments and hedging points.

Based on different values of $c_{ij}(i, j = 1, 2)$, we investigate the optimal policy for the following two cases: (1) $c_{11} \le c_{21}$ and $c_{22} \le c_{12}$, and (2) $c_{11} < c_{21}$ and $c_{12} < c_{22}$. Other cases, such as $c_{11} \ge c_{21}, c_{22} \ge c_{12}$, can be classified into one of the above two cases by simply converting the indices of the two resources. Furthermore, in a special case where $c_{11} = c_{12}$ and $c_{21} = c_{22}$, we can consider the two resources as the same (rescaling if necessary), which is not the interest of this research. Therefore, for the remainder of the paper, we assume at least one of $c_{11} < c_{21}$ and $c_{12} > c_{22}$ is true.

3.2 Case (1): $c_{11} \le c_{21}$ and $c_{22} \le c_{12}$

In this case, R_i is the primary resource used to produce P_i with a lower cost, while the other resource is considered a substitute with a higher production cost. Next, we use the switching curves to characterize the optimal production policy.

Consider $(\tilde{x}^*, \tilde{y}^*)$, which is the hedging point of $\tilde{G}_t(x, y)$. By the definition, it is the optimal solution of $\bar{J}_t(x', y') = \min_{X,Y}\{\tilde{G}_t(X, Y)\}$, the uncapacitated problem of Eq. (6). Hence, $\bar{J}_t^1(\tilde{x}^*, \tilde{y}^*) = \bar{J}_t^2(\tilde{x}^*, \tilde{y}^*) = 0$, and consequently, $\bar{G}_t^1(\tilde{x}^*, \tilde{y}^*) = c_{11} - c_{21} \le 0$ and $\bar{G}_t^2(\tilde{x}^*, \tilde{y}^*) = c_{12} - c_{22} \ge 0$. By the definition of $(\bar{x}^*, \bar{y}^*)$, the hedging point of $\bar{G}_t(x, y)$, we have $\bar{G}_t^1(\bar{x}^*, \bar{y}^*) = \bar{G}_t^2(\bar{x}^*, \bar{y}^*) = 0$. In view of convexity of the cost functions, $\bar{x}^* \ge \tilde{x}^*$ and $\bar{y}^* \le \tilde{y}^*$. Compare the two curves $\bar{S}_1(y)$ and $\tilde{S}_1(y)$. Since $\bar{J}_t^1(\tilde{S}_1(y), y) = 0$ while $\bar{G}_t^1(\bar{S}_1(y), y) = \bar{J}_t^1(\bar{S}_1(y), y) + c_{11} - c_{21} = 0$, we obtain that $\bar{J}_t^1(\bar{S}_1(y), y) = c_{21} - c_{11} \ge 0$, ensuring that $\tilde{S}_1(y) \le \bar{S}_1(y)$. Similarly, we have that $\tilde{S}_0(y) \le \bar{S}_0(y)$ and $\tilde{S}_2(x) \ge \bar{S}_2(x)$. The curves are illustrated in the following figure. Each curve is divided into two parts by its corresponding hedging point. A solid curve represents the part of the curve that will be used to determine the resource allocation and production decisions, while a dashed curve represents the inactive parts in decision processes.

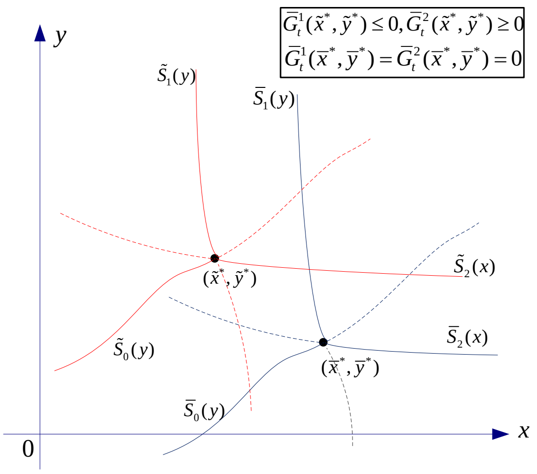

Figure 1. Switching curves for the case $c_{11} \le c_{21}, c_{22} \le c_{12}$.

The six curves provide simple rules to determine the allocation of resources and production decisions. We divide the decision process into two steps, although, as we have mentioned, all the decisions are actually made simultaneously. In the first step, allocation of R_1 is considered. If the initial state (x, y) is on the right side of $\bar{S}_1(y)$ and above $\bar{S}_2(x)$, neither producing P_1 nor P_2 can reduce cost; thus R_1 is not used for production. On the left side of $\bar{S}_1(y) \wedge \bar{S}_0(y)$, producing P_1 is more cost-effective; thus R_1 is used to produce P_1 until R_1 is exhausted or the inventory of P_1 reaches either $\bar{S}_1(y)$ or $\bar{S}_0(y)$. Below the curves $\bar{S}_2(x)$ and

$\bar{S}_0(y)$, only producing P2 is cost-effective, thus R1 is used to produce P2 until R1 is exhausted or the inventory of P2 reaches either $\bar{S}_2(x)$ or $\bar{S}_0(y)$. If the state is on $\bar{S}_0(y)$, producing P1 and P2 has the same cost margin; thus, P1 and P2 are produced simultaneously so the state moves upwards along $\bar{S}_0(y)$ until R1 is exhausted or ($\bar{x}^*, \bar{y}^*$) is reached, above which producing more is not cost-effective. In the second step, R2 is considered. Using the same logic, the production rule can be simply determined by the three curves of $\tilde{G}_t(x, y)$: If the updated state after the first step is on the right side of $\tilde{S}_1(y)$ and above $\tilde{S}_2(x)$, neither producing P1 nor P2 can reduce the cost; thus, R2 is not used for production. On the left side of $\tilde{S}_1(y) \wedge \tilde{S}_0(y)$, producing P1 is more cost-effective, thus R2 is used to produce P1 until R2 is exhausted or the inventory of P1 reaches either $\tilde{S}_1(y)$ or $\tilde{S}_0(y)$. Below the curves $\tilde{S}_2(x)$ and $\tilde{S}_0(y)$, only producing P2 is cost-effective; thus, R2 is used to produce P2 only until R2 is exhausted or the inventory of P2 reaches either $\tilde{S}_2(x)$ or $\tilde{S}_0(y)$. Again, if the state reaches $\tilde{S}_0(y)$ and it is below ($\tilde{x}^*, \tilde{y}^*$), P1 and P2 are produced simultaneously along $\tilde{S}_0(y)$ until ($\tilde{x}^*, \tilde{y}^*$) is reached or R2 is exhausted.

Let $q_{ij}^*(x, y)$ represent the optimal production quantity of Pj by using Ri, $i, j = 1, 2$. The optimal policy has the following properties:

Lemma 2. *(i) If* $\rho_{11}q_{11}^* + \rho_{12}q_{12}^* < M_1$ *and* $\rho_{21}q_{21}^* + \rho_{22}q_{22}^* < M_2$, *then all products are manufactured by their sole primary resource; that is,* $q_{12}^* = q_{21}^* = 0$. *(ii)* $q_{12}^* \times q_{21}^* = 0$. *(iii) If* $c_{12} > c_{22}$, *then* $\tilde{S}_2(\bar{x}^*) - \bar{y}^* \geq M_2 / \rho_{22}$ and $\tilde{S}_2(x) - \bar{S}_2(x) \geq M_2 / \rho_{22}$ *for* $x > \bar{x}^*$.

Property (i) states that if neither of the two resources is exhausted, then all products are produced by their primary resource without using the substitute. This outcome is reasonable since production costs are cheaper when the primary resource is used than when the substitute resource is used. Property (ii) indicates that if one resource primarily used for one product is used to produce the other product, the other resource will not be used to produce this product. Intuitively, if both primary resources are used to produce the other product, then the total cost can be further reduced by using the primary resource to produce more of its own product and less of the other product. Property (iii) ensures that if $c_{12} > c_{22}$ then $\tilde{y}^* - \bar{y}^* \geq \tilde{S}_2(\bar{x}^*) - \bar{y}^* \geq M_2 / \rho_{22}$. The implication is that when the production cost by using the substitute resource is strictly higher than using the primary resource, then the difference of the corresponding inventory levels of the product at the two hedging points is at least the maximal quantity of this product can be produced if only and all of this resource is used. To understand this, consider a special case where $\rho_{ij} = 1$, Property (iii) and the monotonicity of $\tilde{S}_2(x)$ imply that the difference of $\tilde{y}^*$ and $\bar{y}^*$ is not less than M_2.

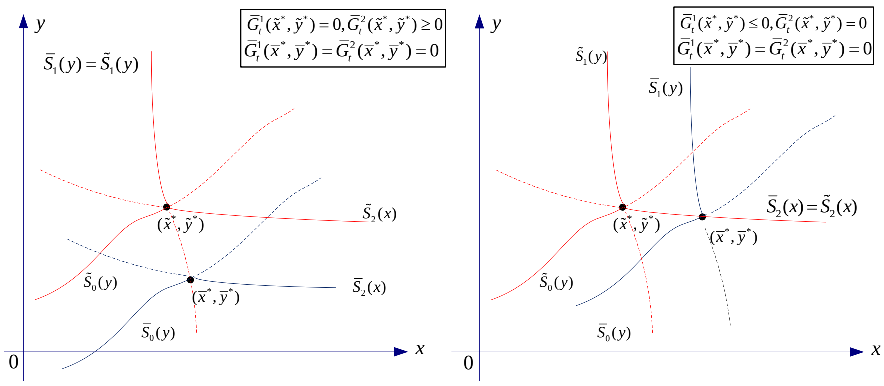

Figure 2. Switching curves for cases (a) $c_{11}=c_{21}$ (left), and (b) $c_{12}=c_{22}$ (right).

There are two special cases: (a) $c_{11}=c_{21}$, and (b) $c_{12}=c_{22}$. In case (a), it is easy to verify that $\bar{S}_1(y)$ and $\tilde{S}_1(y)$ merge to one curve, and in case (b), $\bar{S}_2(x)$ and $\tilde{S}_2(x)$ merge to one curve. The switching curves for the two cases are illustrated in Figure 2. When both $c_{11}=c_{21}$ and $c_{12}=c_{22}$ hold, the corresponding pair curves of $\bar{G}_t(x,y)$ and $\tilde{G}_t(x,y)$ merge and the two hedging points, ($\bar{x}^*$, $\bar{y}^*$) and ($\tilde{x}^*$, $\tilde{y}^*$), overlap. In this special case the optimal policy becomes the same as the one obtained by Chen (2004).

Let (x^*,y^*) be the intersection of $\bar{S}_1(y)$ and $\tilde{S}_2(x)$. We call (x^*,y^*) the global hedging point of the system. By the definitions of the curves, it is easy to check that when $x\ge x^*$ and $y\ge y^*$, it is optimal not to produce. From Lemma 2, we know that when both inventory levels are lower than x^* or y^*, respectively, but R_i is sufficient to produce P_i for $i=1,2$ up to (x^*,y^*), then each R_i is only used to produce P_i up to x^* or y^*, respectively.

Lemma 3 *.(i)* $\dfrac{\bar{x}^*+\tilde{x}^*}{2}\le x^*\le \bar{x}^*$ *and* $\dfrac{\bar{y}^*+\tilde{y}^*}{2}\le y^*\le \tilde{y}^*$; *(ii) For* $y^*-M_2/\rho_{22}\le y\le y^*$, $\bar{S}_1(y)=x^*$.

The property (i) of Lemma 3 provides lower and upper bounds of x^* and y^*, and specifies the approximate location of (x^*,y^*); that is, it should be below but close to $(\bar{x}^*,\tilde{y}^*)$. Property (ii) states that below y^* within the range that y^* is reachable by producing P_2 using R_2 solely, the monotone curve $\bar{S}_1(y)$ becomes a constant x^* within this range. These properties can be derived from the monotonicity of $\bar{S}_1(y)$ and $\tilde{S}_2(x)$.

Based on Lemmas 2 and 3 and the above analysis, we are now ready to characterize the optimal resource allocation and production policy. Our method is based on the following ideas. First, the six switching curves and three hedging points are determined analytically by cost parameters, resource capacities and utilization ratios. Some auxiliary curves, line segments and critical points can be derived accordingly (see

Figure 3). Following the two-step decisions we have introduced (even though they are made simultaneously), we decide the production quantities by using R$_1$, depending on the location of the updated inventory levels to the three switching curves of $\bar{G}_t(x,y)$, then decide the production quantities by using R$_2$, depending on the location of the updated inventory levels to the three switching curves of $\tilde{G}_t(x,y)$. Finally, the coordinate plane of (x,y) can be segmented into eight regions, depending on the different product-resource combinations of the optimal policy, for example, R$_i$ is the only resource used to produce P$_i$, R$_1$ is the only resource used to produce both products, or R$_1$ is the only resource used to produce P$_1$ and P$_2$ is not produced (see Figure 3). Detailed results are presented formally in Theorem 1. Based on our method, optimal decisions can be determined analytically. Mathematical definitions of the regions for this case and subsequent cases can be found in Appendix E. The borders of the regions are highlighted in bold (red). Dashed curves are additional curves that are not boundaries but are important in determining the optimal decisions. Arrow signs are used to describe which resource is used to produce which products. For example, R$_1$ -> P$_1$ & P$_2$ means R$_1$ is used to produce both products. Other arrow signs can be interpreted accordingly.

We also note that the hedging point $(\tilde{x}^*,\tilde{y}^*)$ is not on the boundaries. Rather, another critical point $(\tilde{x}^* - M_1/\rho_{11}, \tilde{y}^*)$, together with the other two hedging points (x^*,y^*) and $(\bar{x}^*,\bar{y}^*)$ are vertices of the regions. We present the optimal policy in the next theorem.

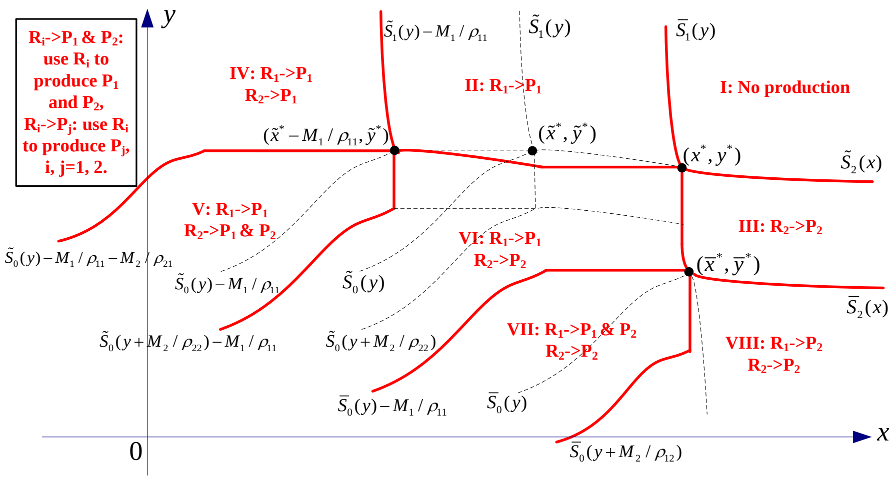

Figure 3. Optimal policy for the case $c_{11} \le c_{21}, c_{22} \le c_{12}$.

Theorem 1. *The optimal allocation and production decisions depend on which of the eight regions the initial state (x,y) is located. Specifically, (1) for $(x,y)\in I$, do not produce; (2) for $(x,y)\in II$, use R$_1$ to produce P$_1$ until R$_1$ is exhausted or up to the base-stock level $\bar{S}_1(y)$; (3) for $(x,y)\in III$, use R$_2$ to produce P$_2$ until R$_2$ is exhausted or up to the base-stock level $\tilde{S}_2(x)$; (4) for $(x,y)\in IV$, use all of R$_1$ and all or part of R$_2$ to produce P$_1$ until R$_2$ is exhausted or P$_1$ is produced up to $\tilde{S}_1(y)$ if $y \ge \tilde{y}^*$, and use all of R$_1$ and R$_2$ to produce*

P_1 if $y < \tilde{y}^*$; (5) for $(x, y) \in V$, use all of R_1 to produce P_1 and all or part of R_2 to produce both products until R_2 is exhausted or the state reaches $(\tilde{x}^*, \tilde{y}^*)$; (6) for $(x, y) \in VI$, use R_1 to produce P_1 until R_1 is exhausted or up to the base-stock level $\bar{S}_1(y)$, use R_2 to produce P_2 until R_2 is exhausted or up to the base-stock level $\tilde{S}_2(x)$; (7) for $(x, y) \in VII$, use R_1 to produce P_1 and P_2 until R_1 is exhausted or the state reaches $(\bar{x}^*, \bar{y}^*)$, and use all of R_2 to produce P_2; and (8) for $(x, y) \in VIII$, use all of R_1 to produce P_2, or part of R_1 to produce P_2 until P_2 is produced up to $\bar{S}_2(x)$ and then use all of R_2 to produce P_2.

Theorem 1 shows how the decision depends on the initial state and the complexity of the optimal structure. Region I represents the area where both inventory levels are high, hence there is no need to produce. In region II, P_2 has sufficient inventory, but P_1 needs more, while it is still not cost-effective to use R_2 to produce P_1 due to the high cost. In region IV, P_2 has sufficient inventory, but P_1's inventory is so low that it is desirable to use R_2 to produce P_1 even after all the R_1 are used to produce P_1. In region V, both products need more inventory, but P_1 is more deficient than P_2, so all R_1 will be used to produce P_1, whereas R_2 will be used to produce both. VI is a more balanced area where both products use the primary resource to produce, and after that either the inventories are sufficient or the resource is exhausted. Regions III, VII and VIII are symmetric to II, IV and V, respectively, so they can be explained similarly.

Some important aspects of Theorem 1 warrant additional comments. First, the left boundaries of regions I, III and VIII are a threshold for producing product 1. We can see from Figure 3 that it is desirable to produce product 1 if the state is on the left side of the boundaries, and not to produce product 1 otherwise. Second, the bottom boundaries of regions I, II and IV are a threshold for producing product 2 (i.e., it is only desirable to produce product 2 when the state is below the boundaries). Third, the lower the inventory level of a product, the more desirable it is to use its primary resource to produce this product, given it is profitable to produce more of this product and the inventory level of the other product is fixed. Fourth, in the regions where it is desirable to produce both products, there are two thresholds that determines whether one resource should be allocated to the production of the other product (i.e., one consists of the boundaries between regions V and VI) and the second one consists of the boundaries between regions VI and VII. Furthermore, both the thresholds are non-decreasing in each inventory level. Fifth, region II and region III are not symmetric, since the left boundary of region II is not $\tilde{S}_1(y)$ but an auxiliary curve $\tilde{S}_1(y) - M_1 / \rho_{11}$. This is because, the curves in Figure 3 correspond to the two production decisions (using two resources, respectively) sequentially. In region II, after production using R_1 the state will be on the right side of $\tilde{S}_1(y)$, ensuring that R_2 will not be used for production. In region III, however, R_1 is not considered, so an auxiliary curve is not necessary. Sixth, in region IV, R_2 is partially (fully) used if the state is on the right (left) side of $\tilde{S}_1(y) - M_1 / \rho_{11} - M_2 / \rho_{21}$. Similarly, in region VIII, R_1 is partially (fully) used if the state is on top of (below) $\bar{S}_2(x) - M_1 / \rho_{12}$.

From Figure 3, we can also see how the optimal decision is affected by production costs, resource capacities and resource utilization ratios. As we have discussed, the distance between $(\tilde{x}^*, \tilde{y}^*)$ and $(\bar{x}^*, \bar{y}^*)$ depends on the production costs. Specifically, $\bar{x}^* - \tilde{x}^*$ is increasing in $c_{21} - c_{11}$ and $\tilde{y}^* - \bar{y}^*$ is increasing in $c_{12} - c_{22}$. Additionally, it is evident that the distances between boundary curves further depend on resource capacities and resource utilization ratios. For example, if M_1 is extremely large, implying that R_1 is uncapacitated, then II and VI will expand to the left areas of the coordinate plane and IV and V will disappear, since there is always sufficient R_1, it is not necessary to use R_2 to produce P_1.

Furthermore, it can also be seen that using substitute resource to produce only occurs in regions V and VII, if the state reaches $\bar{S}_0(y)$ when R_1 is used for production, or $\tilde{S}_0(y)$ when R_2 is used for production. Moreover, it is worth noting that when M_2 / ρ_{22} is sufficiently large, $\tilde{S}_0(y + M_2 / \rho_{22}) - M_1 / \rho_{11}$ might be below $\bar{S}_0(y) - M_1 / \rho_{11}$ or even $\bar{S}_0(y + M_2 / \rho_{12})$. That will cause an overlap of regions V and VII (or even V and VIII). In this case, we can verify whether VII (or VIII) dominates in the overlapped area; thus, we only need to modify the bottom boundary of V while the boundaries of VII and VIII remain the same.

It is worth mentioning that the monotone properties in Lemma 1 are critical to characterize the optimal policy. The assumption $\rho_{11} / \rho_{12} = \rho_{21} / \rho_{22}$ is sufficient to ensure that these monotone properties can be preserved throughout the time horizon. If this assumption is not met, $\rho_{11}^{-1}\bar{G}_t^1(x, y) - \rho_{12}^{-1}\bar{G}_t^2(x, y)$ or $\rho_{21}^{-1}\tilde{G}_t^1(x, y) - \rho_{22}^{-1}\tilde{G}_t^2(x, y)$ may have opposite monotone properties in terms of *x* or *y*. In that case, whether the optimal policy holds or not is unknown.

3.3 Case (2): $c_{11} < c_{21}$ and $c_{12} < c_{22}$

In this case, resource R_1 is more cost-effective than R_2 to produce both products. This setting may arise in a production environment where R_1 is used for regular production while producing with R_2 needs additional technology. Similar to Section 3.2, next we use switching curves to characterize the optimal policy.

We consider ($\tilde{x}^*, \tilde{y}^*$), the hedging point of $\tilde{G}_t(x, y)$. By definition, $\bar{J}_t^1(\tilde{x}^*, \tilde{y}^*) = \bar{J}_t^2(\tilde{x}^*, \tilde{y}^*) = 0$, and consequently, $\bar{G}_t^1(\tilde{x}^*, \tilde{y}^*) = c_{11} - c_{21} < 0$ and $\bar{G}_t^2(\tilde{x}^*, \tilde{y}^*) = c_{12} - c_{22} < 0$. By the definition of ($\bar{x}^*, \bar{y}^*$), the hedging point of $\bar{G}_t(x, y)$, we have $\bar{G}_t^1(\bar{x}^*, \bar{y}^*) = \bar{G}_t^2(\bar{x}^*, \bar{y}^*) = 0$. In view of the cost functions' convexity, $\bar{x}^* > \tilde{x}^*$ and $\bar{y}^* > \tilde{y}^*$. As in Section 3.2, we can obtain $\tilde{S}_1(y) < \bar{S}_1(y)$ and $\tilde{S}_2(x) < \bar{S}_2(x)$. For $\tilde{S}_0(y)$ and $\bar{S}_0(y)$, in view of the definition of $\bar{G}_t(x, y)$ and the assumption $\rho_{11} / \rho_{12} = \rho_{21} / \rho_{22}$, it is easy to verify that $\tilde{S}_0(y) \le \bar{S}_0(y)$ if $\rho_{11}^{-1}(c_{11} - c_{21}) \le \rho_{12}^{-1}(c_{12} - c_{22})$ and $\tilde{S}_0(y) > \bar{S}_0(y)$ otherwise. The curves for the two cases can thus be illustrated in the following figure. Different locations of $\tilde{S}_0(y)$ in terms of $\bar{S}_0(y)$ may change the optimal decision regions. Next we will discuss the two cases in more details.

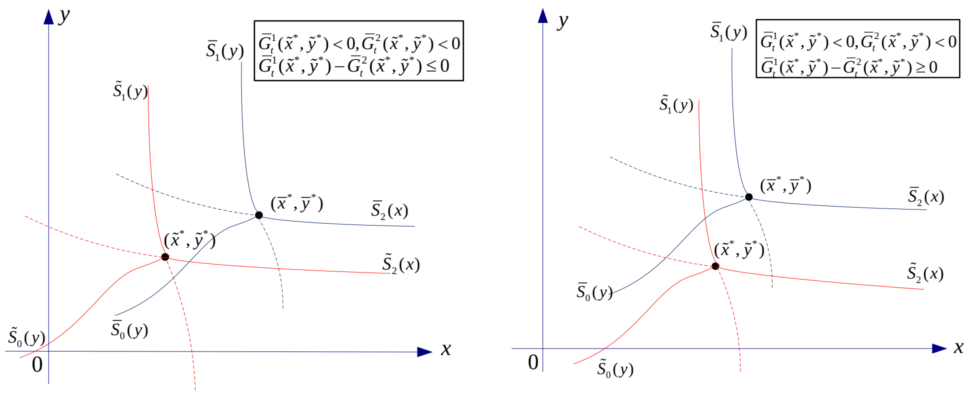

Figure 4. Switching curves for the case $c_{11} < c_{21}, c_{12} < c_{22}$ (left when $\rho_{11}^{-1}(c_{11}-c_{21}) \le \rho_{12}^{-1}(c_{12}-c_{22})$, and right when $\rho_{11}^{-1}(c_{11}-c_{21}) \ge \rho_{12}^{-1}(c_{12}-c_{22})$).

The economic meaning of $\rho_{11}^{-1}(c_{11}-c_{21})$ ($\rho_{12}^{-1}(c_{12}-c_{22})$) is the rescaled difference of production costs for product 1 (product 2) by using the two resources. As illustrated in Figure 4, the locations of $\tilde{S}_0(y)$ and $\bar{S}_0(y)$ depend on the values of $\rho_{11}^{-1}(c_{11}-c_{21})$ and $\rho_{12}^{-1}(c_{12}-c_{22})$. When $\rho_{11}^{-1}(c_{11}-c_{21})$ is larger than $\rho_{12}^{-1}(c_{12}-c_{22})$, it implies that using R_1 to produce P_1 creates more cost reduction than producing P_2. This explains why $\bar{S}_0(y)$ is located on the left of $\tilde{S}_0(y)$, resulting in more P_1 being produced by R_1. To see this, consider a simple example where $\rho_{11} = \rho_{12}$, $c_{11} = 1$, $c_{12} = 2$, $c_{21} = 2$, and $c_{22} = 4$. Due to the high cost of c_{22}, it is desirable to use more R_1 to produce P_2. So at the area where production of P_2 is needed (say below $\bar{S}_0(y)$ and $\tilde{S}_0(y)$), R_1 will be first used to produce P_2 as long as the state does not arrive at $\bar{S}_0(y)$ from the below. Since $\bar{S}_0(y)$ is above or on the left of $\tilde{S}_0(y)$, it ensures that more of P_2 will be produced by R_1. Cases in which $\rho_{11}^{-1}(c_{11}-c_{21})$ is smaller than $\rho_{12}^{-1}(c_{12}-c_{22})$ can be explained similarly.

Despite the curves' different locations, the same decision rules for the allocation of resources and production based on the six curves discussed in Section 3.2 can be applied in this case. Again, let $q_{ij}^*(x,y)$ represent the optimal production quantity of P_j by using R_i, $i, j = 1, 2$. The optimal policy has the following properties:

Lemma 4. *If* $\rho_{11}q_{11}^* + \rho_{12}q_{12}^* < M_1$, *then* $q_{21}^* = q_{22}^* = 0$.

Unlike Section 3.2, in this case it is possible that both q_{12}^* and q_{21}^* are positive. The intuitive reason is that, R_1 is the primary resource used to produce both products, and R_2 will be used for either production if R_1 is not sufficient. Of course, the optimal solutions might not be unique. To illustrate this, consider a special case where ρ_{ij} is identical for all $i, j = 1, 2$, and $c_{21} - c_{11} = c_{22} - c_{12}$. Supposing that the optimal solution $(q_{11}^*, q_{12}^*, q_{21}^*, q_{22}^*)$ satisfies $0 < q_{12}^* \le q_{21}^*$, it is easy to verify that an alternative solution $(q_{11}^* + q_{12}^*, 0, q_{21}^* - q_{12}^*, q_{22}^* + q_{12}^*)$ is also optimal. Therefore, for the remainder of this section, we will discuss only one optimal allocation and production policy although alternative optimal solutions might exist.

According to the values of $\rho_{11}^{-1}(c_{11} - c_{21})$ and $\rho_{12}^{-1}(c_{12} - c_{22})$, the study of the optimal policy can be classified as follows: Case 2.1. $\rho_{11}^{-1}(c_{11} - c_{21}) = \rho_{12}^{-1}(c_{12} - c_{22})$; Case 2.2. $\rho_{11}^{-1}(c_{11} - c_{21}) < \rho_{12}^{-1}(c_{12} - c_{22})$; and Case 2.3. $\rho_{11}^{-1}(c_{11} - c_{21}) > \rho_{12}^{-1}(c_{12} - c_{22})$. The optimal policies for the three cases are similar but have different aspects. For conciseness sake, we next present the results for Case 2.1. As a comparison, the results for Case 2.2 are provided in Appendix C. Case 2.3 is symmetric to Case 2.2, thus is omitted.

3.3.1 Case 2.1: $\rho_{11}^{-1}(c_{11} - c_{21}) = \rho_{12}^{-1}(c_{12} - c_{22})$

When the unit cost of using R_2 (after rescaling by ρ) is the same for the two products, it can be easily verified that $\tilde{S}_0(y) = \bar{S}_0(y)$. The switching curves can thus be shown in the following figure. This is a special case of Figure 4, so can be explained similarly.

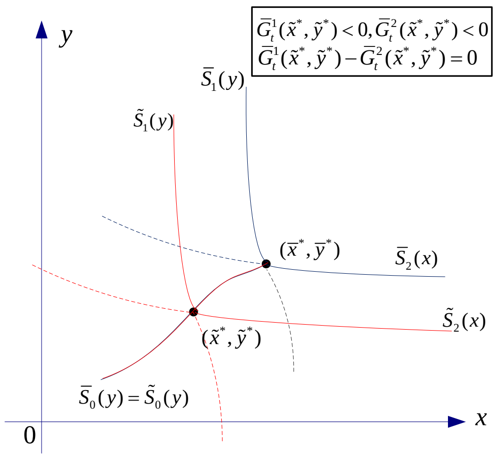

Figure 5. Switching curves for the case $c_{11} < c_{21}, c_{12} < c_{22}$ and $\rho_{11}^{-1}(c_{11} - c_{21}) = \rho_{12}^{-1}(c_{12} - c_{22})$.

We segment the coordinate plane of (x, y) into nine regions based on the optimal decisions made at the state. The segmentation is illustrated in Figure 6. We present the optimal policy in the next theorem.

Theorem 2. *When* $c_{11} < c_{21}, c_{12} < c_{22}$ *and* $\rho_{11}^{-1}(c_{11} - c_{21}) = \rho_{12}^{-1}(c_{12} - c_{22})$ *, the optimal allocation and production decisions depend on which of the nine regions the initial state* (x, y) *is in. Specifically, (1) for* $(x, y) \in I$ *, the optimal decision is not to produce; (2) for* $(x, y) \in II$ *, use* R_1 *to produce* P_1 *until* R_1 *is exhausted or produced up to the base-stock level* $\bar{S}_1(y)$*; (3) for* $(x, y) \in III$ *, use* R_1 *to produce* P_1 *(*P_2*) first if* $x < \bar{S}_0(y)$ *(*$x > \bar{S}_0(y)$*), then produce both and update the state along* $\bar{S}_0(y)$ *until* R_1 *is exhausted or the state reaches* $(\bar{x}^*, \bar{y}^*)$*; (4) for* $(x, y) \in IV$ *, use* R_2 *to produce* P_2 *until* R_2 *is exhausted or produced up to the base-stock level* $\bar{S}_2(x)$*; (5) for* $(x, y) \in V$ *, use all of* R_1 *to produce* P_1 *and all or part of* R_2 *to produce* P_1 *until* R_2 *is exhausted or* P_1 *is produced up to the base-stock level* $\tilde{S}_1(y)$ *if* $y > \tilde{y}^*$*, or use all of* R_1 *and* R_2 *to produce* P_1 *if* $y \le \tilde{y}^*$*; (6) for* $(x, y) \in VI$ *, use all of* R_1 *to produce* P_1*, use* R_2 *to produce both products until* R_2 *is exhausted or the state reaches* $(\tilde{x}^*, \tilde{y}^*)$*; (7) for* $(x, y) \in VII$ *, use all* R_1 *to produce* P_1 *and* P_2*, and use all or part of* R_2 *to produce both products until* R_2 *is exhausted or the state reaches* $(\tilde{x}^*, \tilde{y}^*)$*; (8) for* $(x, y) \in VIII$ *, use all of* R_1 *to produce* P_2*, and all or part of* R_2 *to produce both products until* R_2 *is exhausted or the state reaches* $(\tilde{x}^*, \tilde{y}^*)$*; (9) for* $(x, y) \in IX$ *, use all of* R_1*, all or part of* R_2 *to produce* P_2 *until* R_2 *is exhausted or up to the base-stock level* $\bar{S}_2(x)$.

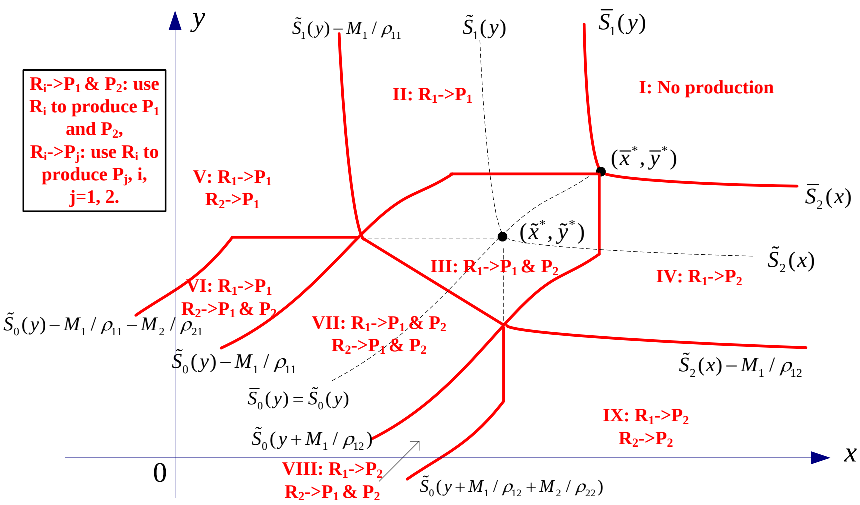

Figure 6. Optimal policy for the case $c_{11} < c_{21}, c_{12} < c_{22}$ and $\rho_{11}^{-1}(c_{11} - c_{21}) = \rho_{12}^{-1}(c_{12} - c_{22})$.

Four points are worth mentioning. First, for all the cases in Theorem 2, using one resource to produce both products (states in regions III, VI, VII and VIII) only occurs when the state reaches $\bar{S}_0(y)$. The simple rule always holds when the state is not on $\bar{S}_0(y)$: if production is desirable, then produce P_1 (P_2) only if $x < \bar{S}_0(y)$ ($x > \bar{S}_0(y)$). Second, similar to Section 3.2, there exist thresholds that determine whether to produce product 1 (i.e., the left boundaries of regions I, IV and IX), and whether to produce product 2 (i.e., the bottom boundaries of regions I, II and V). Third, because R_1 is more cost-effective than R_2 to produce both products, it is always beneficial to give priority to R_1 and consider R_2 only if R_1 is exhausted. We can see that it is possible here but is impossible for the case $c_{11} \le c_{21}$ and $c_{22} \le c_{12}$ shown in Figure 3, to use both R_i to produce P_j, $i, j = 1, 2, i \neq j$. Additionally, in Figure 6, there is no region in which R_2 is only used for production, but this may occur in Figure 3. Fourth, for the same reason, the hedging points (x^*, y^*) appeared in Figure 3 has merged with $(\bar{x}^*, \bar{y}^*)$; thus in this case there are only two hedging points: $(\bar{x}^*, \bar{y}^*)$ and $(\tilde{x}^*, \tilde{y}^*)$.

In summary, our analysis of different cases shows that the optimal production and resource allocation policy depends significantly on cost parameters and resource utilization ratios. However, their impact on the optimal decisions has been underemphasized or ignored in the literature due to the technical complexity. Our results also show that although we could derive analytical solutions for two-resource, two-product systems, the optimal policy is too complicated to be implemented. It might be interesting to find implementable heuristics that are close to optimal.

4 Heuristic Policies

As we can see in Section 3, the optimal policy is very complicated, so implementing it in practice is challenging. Therefore, a natural question to ask from a managerial viewpoint is whether a simpler heuristic policy is acceptable for practical purposes. To address this question, we perform a series of numerical studies where the total cost in various policies is compared. Using the optimal policy as a benchmark, we will evaluate three heuristic policies: H1, H2 and H3. H1 is a policy without resource flexibility**,** and thus the gap between it and the optimal policy represents the value of resource flexibility. Similar heuristic policies have been used in the literature (see, e.g., Bish et al., 2005). H2 is a heuristic based on the analytical results we obtained in Section 3 but we simplify some rules so the computational complexity could be reduced. H3 is a myopic policy based on optimal decisions for a one-period problem. The purpose of studying H3 is to demostrate the gap between a static policy and the optimal dynamic policy. Specifically, they are defined as follows:

H1: Modified base-stock policy (which follows a base-stock policy when possible, and when the prescribed production quantity would exceed the capacity, produce the capacity) for an inflexible system, in which there is no resource flexibility; (i.e., each resource can only produce one product) and thus, the two products could separate and each product follows a modified base-stock policy.

H2: This heuristic is described as follows: We first assume there is no flexibility in the system, and calculate the optimal base-stock levels as in H1) (suppose they are $\hat{x}^*$ and $\hat{y}^*$, respectively). Consider two cases:

- If $c_{11} \le c_{21}, c_{22} \le c_{12}$, (1) use R_i to produce P_i , $i = 1,2$, based on a modified base-stock policy: produce P_1 (P_2) up to $\hat{x}^*$ ($\hat{y}^*$) if $\hat{x}^* - M_1 / \rho_{11} < x \le \hat{x}^*$ ($\hat{y}^* - M_2 / \rho_{22} < y \le \hat{y}^*$), use all of R_1 (R_2) to produce P_1 (P_2) if $x \le \hat{x}^* - M_1 / \rho_{11}$ ($y \le \hat{y}^* - M_2 / \rho_{22}$), and do not produce P_1 (P_2) if $x > \hat{x}^*$ ($y > \hat{y}^*$); (2) If the inventory level of P_1 (P_2) does not reach its base-stock level $\hat{x}^*$ ($\hat{y}^*$), check if there is still R_2 (R_1) available. If so use the unused R_2 (R_1) to produce P_1 (P_2) until it reaches its base-stock level or as close as possible.
- If $c_{11} < c_{21}, c_{12} < c_{22}$, (1) use R_1 to produce both products simultaneously. Each is based on a modified base-stock policy: the production of P_1 (P_2) continues until its inventory level reaches $\hat{x}^*$ ($\hat{y}^*$) or R_1 is exhausted. (2) use R_2 to continue production until each product reaches its base-stock level or R_2 is exhausted.

H3: We calculate the optimal decisions for a one-period problem, and then repeat the optimal decisions in each period.

Note that both H1 and H2 simplify the computation dramatically in the way that the base-stock of each product is obtained from H1, which is now state-independent. Additionally, in the second case of H2, an allocation of which product to be produced is also needed. We simplify this rule by producing both products simultaneously if both inventory levels are lower than their base-stock levels and resource is available. H3 simplifies the computational complexity by only calculating optimal decisions for one period then follows the same decisions in each period.

To compare the described policies, we compute the percentage difference in cost between the optimal solution and each heuristic policy for each state, taking the maximum and average values, respectively, over all of the state space, which is truncated to $\{-10 \le x \le 10, -10 \le y \le 10\}$. We use the following experiments to test the heuristics. For all our examples, we assume that $T = 10$ and $\alpha = 0.95$. We test the following two groups of distributions. (I) D_1 and D_2 are discrete and uniformly distributed between 1 and 10; (II) D_1 and D_2 are approximately normally distributed with a mean of 5 and variance of 4. Here, 'approximately' means the normal distribution is discretized by only considering integer values, and is truncated to ensure the demand is non-negative. In each group, we consider different values of h_i, b_i, c_{ij} and M_k / ρ_{ij} $(i, j, k = 1,2)$, and then we vary each parameter value and analyze the impact of the change on policy effectiveness.

Furthermore, to make the optimal decisions in different periods more stationary, we revise the terminal function by assuming there is a penalty cost if the inventory level of a product is negative in period $T+1$ or there is a salvage value if the inventory level is positive. Specifically, we change $f_{T+1}(x, y) = 0$ to

$f_{T+1}(x,y) = -(c_{11} \wedge c_{21})x - (c_{12} \wedge c_{22})y$ where $a \wedge b = \min(a,b)$. Since it is unknown if a leftover is produced by which resource or a backorder will be filled by which resource, we assume the salvage value or cost is based on the smaller product cost of the two resources. It is worth noting this change does not affect the theoretical analysis and structural properties we obtained in Section 3.

For case (1) where $c_{11} \le c_{21}, c_{22} \le c_{12}$, the numerical results for distributions I and II are reported in Tables 2 and 3, respectively. Results in each table are classified into three sub-groups, depending on the parameter values. Specifically, we only change c_{ij} $(i \ne j)$ in group 1, M_k / ρ_{ij} in group 2, h_i, or b_i in group 3.

Table 2. Distributions I for case $c_{11} \le c_{21}, c_{22} \le c_{12}$.

Parameters												Error					
												H1		H2		H3	
h_1	h_2	b_1	b_2	c_{11}	c_{12}	c_{21}	c_{22}	$\frac{M_1}{\rho_{11}}$	$\frac{M_1}{\rho_{12}}$	$\frac{M_2}{\rho_{21}}$	$\frac{M_2}{\rho_{22}}$	Max	Average	Max	Average	Max	Average
0.1	0.1	0.3	0.3	0.5	0.5	0.5	0.5	6	6	6	6	25%	8%	7%	2%	7%	1%
0.1	0.1	0.3	0.3	0.5	0.6	0.6	0.5	6	6	6	6	21%	6%	5%	2%	6%	2%
0.1	0.1	0.3	0.3	0.5	0.7	0.7	0.5	6	6	6	6	19%	5%	5%	2%	6%	3%
0.1	0.1	0.3	0.3	0.5	0.8	0.8	0.5	6	6	6	6	16%	4%	5%	2%	18%	5%
0.1	0.1	0.3	0.3	0.5	0.9	0.9	0.5	6	6	6	6	14%	3%	5%	3%	16%	4%
0.1	0.1	0.3	0.3	0.5	1.0	1.0	0.5	6	6	6	6	13%	3%	7%	3%	14%	4%
0.1	0.1	0.3	0.3	0.5	0.5	0.5	0.5	8	8	4	4	61%	23%	9%	4%	9%	2%
0.1	0.1	0.3	0.3	0.5	0.6	0.6	0.5	8	8	4	4	55%	20%	7%	4%	7%	2%
0.1	0.1	0.3	0.3	0.5	0.7	0.7	0.5	8	8	4	4	49%	17%	9%	4%	9%	5%
0.1	0.1	0.3	0.3	0.5	0.8	0.8	0.5	8	8	4	4	44%	15%	11%	5%	41%	16%
0.1	0.1	0.3	0.3	0.5	0.9	0.9	0.5	8	8	4	4	39%	13%	12%	5%	36%	14%
0.1	0.1	0.3	0.3	0.5	1.0	1.0	0.5	8	8	4	4	35%	11%	14%	6%	32%	12%
0.15	0.1	0.3	0.3	0.5	0.7	0.7	0.5	6	6	6	6	20%	7%	4%	1%	4%	1%
0.2	0.1	0.3	0.3	0.5	0.7	0.7	0.5	6	6	6	6	18%	5%	5%	2%	5%	2%
0.25	0.1	0.3	0.3	0.5	0.7	0.7	0.5	6	6	6	6	18%	5%	5%	2%	6%	3%
0.1	0.1	0.35	0.3	0.5	0.7	0.7	0.5	6	6	6	6	23%	5%	7%	2%	7%	3%
0.1	0.1	0.4	0.3	0.5	0.7	0.7	0.5	6	6	6	6	27%	6%	9%	2%	9%	3%
0.1	0.1	0.45	0.3	0.5	0.7	0.7	0.5	6	6	6	6	32%	7%	11%	3%	11%	3%

We see that for all of our numerical examples in Table 2, the cost gaps between the optimal policy and H1 are substantial: the maximum value ranges from 13% to 61% and the average value over the possible states ranges from 3% to 23%. This implies that resource flexibility could dramatically decrease the production cost. For all the examples H2 could improve the performance significantly. For cases with identical M_k / ρ_{ij} (groups 1 and 3), the maximum difference ranges from 4% to 11% and the average difference over the possible states ranges is within 3%; for cases with non-identical M_k / ρ_{ij} (group 2), the gap is larger, with the maximum difference ranging from 7% to 14% and the average difference over the possible states ranges is within 6%. It is evident that although a base-stock policy is common in practice, it is not efficient in systems with multiple flexible resources, and developing more sophisticated heuristics is necessary. In most of the cases, the performance of H3 is similar to that of H2. In several cases H3 performs even slightly better than H2 (when $c_{ij}(i, j = 1, 2)$ have identical or similar values). However, in cases when

c_{ij} have different values, the performance of H3 is much worse (it may be worse than H1), implying that although a static policy like H3 could perform well (compared to H2), H2 is more robust in general.

There are some interesting observations. First, H1 performs better when $c_{ij}\,(i \neq j)$ is much larger than c_{ii}. In this case it is more desirable to use its primary resource to produce each product. Thus, a separate base-stock policy without pooling is closer to the optimal policy. However, H2 shows the opposite pattern. For example, in the second sub-group in Table 2, the average gap increases from 4% to 6% when c_{12} and c_{21} increase from 0.5 to 1.0. This observation is reasonable because a large disparity between c_{ij} and c_{ii} implies that the cost of resource pooling is also high. With the increase of $c_{ij}\,(i \neq j)$, H3's performance first decreases, but slightly increases when c_{ij} increases further. Second, for most of the numerical examples, the performance in sub-group 1 is better than the corresponding average performance in sub-group 2. Third, the gap between H1 and the optimal policy is more sensitive to the backordering cost than to the holding cost. The higher the backorder cost, the larger the gap is, implying flexibility from resource pooling is more important.

Table 3. Distributions II for case $c_{11} \leq c_{21}, c_{22} \leq c_{12}$.

Parameters												Error					
												H1		H2		H3	
h_1	h_2	b_1	b_2	c_{11}	c_{12}	c_{21}	c_{22}	$\frac{M_1}{\rho_{11}}$	$\frac{M_1}{\rho_{12}}$	$\frac{M_2}{\rho_{21}}$	$\frac{M_2}{\rho_{22}}$	Max	Average	Max	Average	Max	Average
0.1	0.1	0.3	0.3	0.5	0.5	0.5	0.5	6	6	6	6	26%	9%	5%	1%	5%	0%
0.1	0.1	0.3	0.3	0.5	0.6	0.6	0.5	6	6	6	6	23%	8%	4%	1%	4%	0%
0.1	0.1	0.3	0.3	0.5	0.7	0.7	0.5	6	6	6	6	20%	7%	4%	1%	4%	1%
0.1	0.1	0.3	0.3	0.5	0.8	0.8	0.5	6	6	6	6	18%	6%	3%	2%	14%	2%
0.1	0.1	0.3	0.3	0.5	0.9	0.9	0.5	6	6	6	6	16%	6%	4%	2%	12%	2%
0.1	0.1	0.3	0.3	0.5	1.0	1.0	0.5	6	6	6	6	14%	5%	5%	3%	10%	1%
0.1	0.1	0.3	0.3	0.5	0.5	0.5	0.5	8	8	4	4	70%	27%	9%	4%	9%	1%
0.1	0.1	0.3	0.3	0.5	0.6	0.6	0.5	8	8	4	4	63%	24%	8%	4%	7%	1%
0.1	0.1	0.3	0.3	0.5	0.7	0.7	0.5	8	8	4	4	56%	21%	8%	4%	6%	3%
0.1	0.1	0.3	0.3	0.5	0.8	0.8	0.5	8	8	4	4	51%	19%	10%	5%	44%	15%
0.1	0.1	0.3	0.3	0.5	0.9	0.9	0.5	8	8	4	4	46%	17%	12%	6%	39%	13%
0.1	0.1	0.3	0.3	0.5	1.0	1.0	0.5	8	8	4	4	41%	15%	13%	6%	36%	12%
0.15	0.1	0.3	0.3	0.5	0.7	0.7	0.5	6	6	6	6	20%	7%	4%	1%	4%	1%
0.2	0.1	0.3	0.3	0.5	0.7	0.7	0.5	6	6	6	6	20%	7%	3%	1%	3%	1%
0.25	0.1	0.3	0.3	0.5	0.7	0.7	0.5	6	6	6	6	20%	7%	3%	1%	3%	1%
0.1	0.1	0.35	0.3	0.5	0.7	0.7	0.5	6	6	6	6	24%	8%	5%	1%	5%	1%
0.1	0.1	0.4	0.3	0.5	0.7	0.7	0.5	6	6	6	6	28%	8%	6%	1%	6%	1%
0.1	0.1	0.45	0.3	0.5	0.7	0.7	0.5	6	6	6	6	32%	9%	8%	2%	8%	1%

The numerical results for distribution II of Case $c_{11} \leq c_{21}, c_{22} \leq c_{12}$ are reported in Table 3. Similar to distribution I, we see that for all of our numerical examples, the cost gaps between the optimal policy and H1 are still substantial: the maximum value ranges from 14% to 70% and the average value over the possible states ranges from 5% to 27%. H2 and H3 show similar patterns as of distribution I. The performance of H3 is quite good except for the cases when $c_{ij}\,(i \neq j)$ is much larger than c_{ii} in sub-group 2. In Table 2 and 3, the performance of H3 gets much worse when c_{ij} increases to 0.8. The possible reason is, since H3 is based on a single-period problem, once c_{ij} is 0.8 or larger, using resource *i* to produce product *j* incurs a high production cost that cannot cover the backordering cost reduction (noting in this case $c_{ij} - c_{ii} = 0.3$ and $b_{ii} = 0.3$); thus, the optimal decision will be not to use resource *i* to produce product *j*. However, for multi-period problems, the optimal decision might be quite different, since using resource *i* to produce product *j* is more desirable to cover backordering costs for multiple periods. This explains why the performance of H3 starts getting much worse when c_{ij} increases to 0.8 but may not be worse when we further increase c_{ij}.

For case (2) where $c_{11} < c_{21}, c_{12} < c_{22}$, the numerical results for distributions I and II are reported in Tables 4 and 5, respectively. We can see that the performance of H2 is worse than that in the case $c_{11} \le c_{21}$ and $c_{22} \le c_{12}$; now, the maximum gap for distribution I ranges from 8% to 38% and the average gap over the possible states ranges from 4% to 10%. The underlying reason is that in Case 2 we need to allocate each type of resource between the two products, which is not the case for Case 1. To simplify the allocation rule, we balance the two products based on their inventory levels and order-up-to levels. This allocation will cause more errors between the optimal policy and H2. In Case 1 the primary resource has clear priority over the substitute resource, thus this allocation rule is not needed.

H3 may work well when $c_{ij}\,(i \ne j)$ and c_{ii} have similar values. However, in some cases especially when $c_{ij}\,(i \ne j)$ is much larger than c_{ii}, H3 performs even worse than that in Case 1. The possible reason is, in Case 2 one resource is prioritized to produce both products, so c_{ij} is more used to calculate the total expected cost. Since H3 is based on a single-period problem, when c_{ij} is larger (compared to c_{ii}), H3 tries to avoid using resource *i* to produce product *j*. This makes H3 deviate from the optimal policy. In cases where $c_{ij}=1$, H3 even performs much worse than H1.

Table 4. Distributions I for the case $c_{11} < c_{21}, c_{12} < c_{22}$.

Parameters												Error					
h_1	h_2	b_1	b_2	c_{11}	c_{12}	c_{21}	c_{22}	$\frac{M_1}{\rho_{11}}$	$\frac{M_1}{\rho_{12}}$	$\frac{M_2}{\rho_{21}}$	$\frac{M_2}{\rho_{22}}$	H1		H2		H3	
												Max	Average	Max	Average	Max	Average
0.1	0.1	0.3	0.3	0.5	0.5	0.6	0.6	6	6	6	6	25%	8%	11%	5%	11%	3%
0.1	0.1	0.3	0.3	0.5	0.5	0.7	0.7	6	6	6	6	25%	8%	11%	5%	10%	7%
0.1	0.1	0.3	0.3	0.5	0.5	0.8	0.8	6	6	6	6	26%	19%	11%	5%	31%	21%
0.1	0.1	0.3	0.3	0.5	0.5	0.9	0.9	6	6	6	6	27%	10%	11%	6%	28%	18%
0.1	0.1	0.3	0.3	0.5	0.5	1.0	1.0	6	6	6	6	28%	11%	12%	7%	51%	20%
0.1	0.1	0.3	0.3	0.5	0.5	0.6	0.6	8	8	4	4	63%	25%	19%	7%	19%	3%
0.1	0.1	0.3	0.3	0.5	0.5	0.7	0.7	8	8	4	4	64%	27%	38%	7%	38%	22%
0.1	0.1	0.3	0.3	0.5	0.5	0.8	0.8	8	8	4	4	66%	30%	38%	8%	33%	22%
0.1	0.1	0.3	0.3	0.5	0.5	0.9	0.9	8	8	4	4	68%	32%	38%	9%	33%	23%
0.1	0.1	0.3	0.3	0.5	0.5	1.0	1.0	8	8	4	4	70%	35%	21%	10%	93%	44%
0.15	0.1	0.3	0.3	0.5	0.7	0.7	0.5	8	8	4	4	48%	17%	9%	4%	8%	5%
0.2	0.1	0.3	0.3	0.5	0.7	0.7	0.5	8	8	4	4	48%	17%	9%	4%	8%	5%
0.25	0.1	0.3	0.3	0.5	0.7	0.7	0.5	8	8	4	4	47%	17%	9%	4%	9%	5%
0.1	0.1	0.35	0.3	0.5	0.7	0.7	0.5	8	8	4	4	48%	17%	9%	4%	8%	5%
0.1	0.1	0.4	0.3	0.5	0.7	0.7	0.5	8	8	4	4	48%	16%	9%	4%	9%	4%
0.1	0.1	0.45	0.3	0.5	0.7	0.7	0.5	8	8	4	4	47%	16%	8%	4%	9%	5%

In the first sub-group, we increase production costs by changing R_2 from 0.6 to 1.0. Unlike that in Tables 2 and 3, here the performance of H1 is not sensitive to this change. This insensitivity might be due to R_2 only being used if R_1 is insufficient for the production requirement. The average performance of H2 is slightly increased in production costs by using R_2. Comparing the first and second sub-groups, for most of the cases the average gaps for both heuristics in the second sub-groups are slightly higher. From the third sub-group, we can again see that H1 is more sensitive to the backordering cost.

In summary, in all our numerical experiments there are big gaps between the individual base-stock policy and the optimal policy, implying that the individual base-stock policy is inefficient in systems with multiple flexible resources. In particular, the value of resource flexibility is more significant when the resource is limited. When resources have sufficiently large quantities, we can expect the gap to be much smaller. A myopic policy based on single-period optimal decisions works well when unit production costs using different resources are similar. However, it may perform poorly when production costs using different resources are significantly different; and may perform worse when one resource is prioritized to produce both products. A more sophisticated but practical heuristic is developed. It performs very well in case $c_{11} \le c_{21}, c_{22} \le c_{12}$. In case $c_{11} < c_{21}, c_{12} < c_{22}$, it could reduce the gap dramatically, but still has space to improve. There is a tradeoff between performance and implementation complexity.

Table 5. Distributions II for the case $c_{11} < c_{21}, c_{12} < c_{22}$.

Parameters												Error					
h_1	h_2	b_1	b_2	c_{11}	c_{12}	c_{21}	c_{22}	$\frac{M_1}{\rho_{11}}$	$\frac{M_1}{\rho_{12}}$	$\frac{M_2}{\rho_{21}}$	$\frac{M_2}{\rho_{22}}$	H1		H2		H3	
												Max	Average	Max	Average	Max	Average
0.1	0.1	0.3	0.3	0.5	0.5	0.6	0.6	6	6	6	6	27%	10%	10%	4%	4%	1%
0.1	0.1	0.3	0.3	0.5	0.5	0.7	0.7	6	6	6	6	29%	12%	10%	4%	9%	4%
0.1	0.1	0.3	0.3	0.5	0.5	0.8	0.8	6	6	6	6	31%	13%	10%	5%	37%	26%
0.1	0.1	0.3	0.3	0.5	0.5	0.9	0.9	6	6	6	6	32%	14%	10%	6%	33%	23%
0.1	0.1	0.3	0.3	0.5	0.5	1.0	1.0	6	6	6	6	34%	16%	12%	7%	69%	28%
0.1	0.1	0.3	0.3	0.5	0.5	0.6	0.6	8	8	4	4	73%	31%	18%	9%	18%	2%
0.1	0.1	0.3	0.3	0.5	0.5	0.7	0.7	8	8	4	4	76%	34%	18%	9%	18%	2%
0.1	0.1	0.3	0.3	0.5	0.5	0.8	0.8	8	8	4	4	79%	37%	18%	11%	41%	21%
0.1	0.1	0.3	0.3	0.5	0.5	0.9	0.9	8	8	4	4	83%	41%	21%	12%	43%	25%
0.1	0.1	0.3	0.3	0.5	0.5	1.0	1.0	8	8	4	4	86%	45%	24%	13%	124%	61%
0.15	0.1	0.3	0.3	0.5	0.7	0.7	0.5	8	8	4	4	55%	21%	9%	5%	6%	3%
0.2	0.1	0.3	0.3	0.5	0.7	0.7	0.5	8	8	4	4	55%	21%	8%	4%	6%	3%
0.25	0.1	0.3	0.3	0.5	0.7	0.7	0.5	8	8	4	4	55%	21%	8%	4%	6%	3%
0.1	0.1	0.35	0.3	0.5	0.7	0.7	0.5	8	8	4	4	56%	21%	9%	4%	7%	3%
0.1	0.1	0.4	0.3	0.5	0.7	0.7	0.5	8	8	4	4	56%	21%	9%	4%	7%	3%
0.1	0.1	0.45	0.3	0.5	0.7	0.7	0.5	8	8	4	4	56%	21%	9%	4%	8%	3%

5 Discussions and Extensions

This section discusses partial resource flexibility to only one resource and extends our model to two different cases with multiple resources.

5.1 Flexibility with Only One Resource

Consider a special case where R_1 can be used to produce both products while R_2 can only be used to produce P_2. This setting may arise in an environment where P_1 can be produced by only a particular resource while P_2 accepts both resources.

If $c_{11} \le c_{21}, c_{22} \le c_{12}$, it is equivalent to assuming that $c_{21} = +\infty$. Hence, the results shown in Figures 2 and 3 still hold. The difference is since c_{21} is infinitely large, $\tilde{S}_1(y)$ and $\tilde{S}_0(y)$ become infinitely small. Therefore, $\tilde{S}_1(y)$ and $\tilde{S}_0(y)$ will disappear from Figure 2. Accordingly, for the optimal policy shown in Figure 3, regions IV and V will disappear and the void will be filled by regions II and VI, while the other regions remain similar shapes.

For Case $c_{11} < c_{21}, c_{12} < c_{22}$ and $\rho_{11}^{-1}(c_{11} - c_{21}) < \rho_{12}^{-1}(c_{12} - c_{22})$, we refer to Figure 4 (left) for the switching curves and Figure A2 (see Appendix C) for the optimal policy. In Figure 4 (left), $\tilde{S}_1(y)$ and $\tilde{S}_0(y)$ become infinitely small, thus they will disappear from the figure. We can also characterize the optimal policy by modifying Figure A2 in the following way: regions II, V, VI and VII merge to one region, and in region VIII, R_2 is only used to produce P_2. All the others are the same as described in Corollary 1.

A more special case is that each R_i can only produce P_i, ($i = 1, 2$). In such a setting, we can apply the results in Appendix A. Only $\bar{S}_1(y)$ and $\tilde{S}_2(x)$ appear in Figure A1. Since the two productions are produced separately, the two curves become two lines. The optimal policy for each product is a base-stock policy.

5.2 Production with Multiple Flexible and Inflexible Resources

Consider a system that uses three types of resources (R_i, $i = 0, 1, 2$) to produce two products (P_i, $i = 1, 2$). R_i can be used to only produce P_i, $i = 1, 2$, while R_0 is flexible and can be used to produce both products; see Figure 7. Van Mieghem (1998) studies a similar system but focuses on the optimal investment in flexible manufacturing capacity. Let ρ_{ij} denote the amount of R_i that is needed to produce one unit of P_j and M_i the available resource of R_i for $i = 0, 1, 2$ and $j = 1, 2$. As in Section 3, we assume that $\rho_{11} / \rho_{22} = \rho_{01} / \rho_{02}$. All other notations are the same as in Section 2. We can formulate the problem by revising Eq. (5) and (6) as

$$\bar{f}_t(x, y) = \min_{\substack{0 \le \rho_{11}(x'-x) \le M_1 \\ 0 \le \rho_{22}(y'-y) \le M_2}} \{\bar{G}_t(x', y')\},$$

and

$$\bar{J}_t(x', y') = \min_{\substack{\rho_{01}(X-x') + \rho_{02}(Y-y') \le M_0 \\ X \ge x', Y \ge y'}} \{\tilde{G}_t(X, Y)\}.$$

Note the only difference between these equations and (5) and (6) is their constraints. Following the approach in Section 3, it can be shown that with the revised constraints, all the cost functions are ρ -

differential monotone. Therefore, the optimal policy can be characterized by the switching curves, and the results obtained in Section 3 hold.

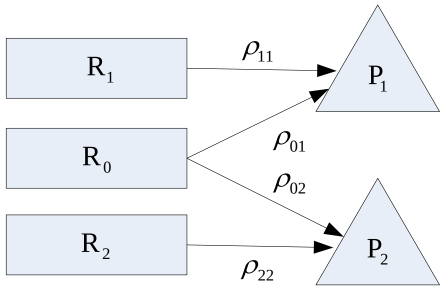

Figure 7. A two-resource, two-product flexible system

This model can be also applied to a flexible manufacturing system. R_i, $i = 1, 2$, is considered a machine without flexibility, and R_0 is a machine with flexibility. ρ_{ij} represents the capacity of machine *i* to produce product *j*.

5.3 Systems with Multiple Flexible Resources

Consider the general case when multiple flexible resources can be used to produce two products. Suppose there are *n* types of resources, which are denoted by R_i, $i = 1, 2, \ldots n$. The available amount of R_i in each period is M_i. Each R_i can be used to produce P_1 and P_2. ρ_{ij} of R_i is needed to produce one unit of P_j, $j = 1, 2$. ρ_{ij} might be different for the same *j* and different *i*, although the ratio ρ_{i1} / ρ_{i2} is the same for all $i = 1, 2, \ldots n$. c_{ij} is the unit production cost if R_i is used to produce P_j. Similarly, as in Section 2, we can formulate the problem sequentially. In the first step we decide how much of R_1 is used to produce P_1 and P_2, assuming the optimal decisions for the other resources are embedded. Then we consider the optimal decision control for R_2 assuming the optimal decisions for R_i, $i = 3, \ldots n$ have been made, and so on until the optimal control for R_n. The optimal control for each R_i has the same format as Eq. (7). Therefore, there exist a hedging point and three switching curves for each R_i, $i = 1, \ldots n$. The curves have monotone properties as we have discussed. The locations of hedging points and switching curves for different resources are dependent on the cost c_{ij}. To characterize the optimal policy, we can just repeatedly apply the decision rules based on the three switching curves for each R_i. However, the segmentation of the decision regions will be much more complicated compared with that of the two-resource case presented in Section 3.

6 Conclusions

This paper studies a finite horizon, periodic review production system that uses flexible resources to produce two products with stochastic demands. Each resource has a limited capacity and can be used to produce both products with a resource/product specified production rate. We investigate the optimal production quantity for each product and the optimal policy to allocate the limited resources dynamically between the products. We show that cost functions are ρ - differential monotone, which is helpful to characterize the optimal production and allocation policy. The optimal policy can be characterized by hedging points and state-dependent switching curves. Comparing the optimal solution to heuristic policies reveals that the resource flexibility creates substantial value in the production system.

There are several possible directions for future research. First, we consider systems with two products. A more general and more challenging problem is to study whether similar structural properties and optimal policy can be extended to cases with multiple products. Second, to simplify our analysis, we have assumed that there is no production lead time. It will be interesting to relax this assumption and extend our work to the case with lead times. Third, we proposed a heuristic for multi-resource and two-product systems without considering the impact of the resource utilization ratio. It will be interesting to develop a more sophisticated policy and use numerical studies to test different scenarios in details. Finally, the two-resource system we study in this paper is just a special case of the more complicated flexible manufacturing systems mentioned in the literature, for example, Graves and Tomlin (2003), and Chou et al. (2010). Due to technical complexity, the optimal allocation and production rules are still unclear. It will be interesting to investigate the optimal rules for systems with these flexible structures.

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