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Buyer Financing in Pull Supply Chains: Zero-Interest Early Payment or In-House Factoring?

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Abstract

This paper investigates the efficacy of zero-interest early payment financing (alternatively referred to as early payment) and positive-interest in-house factoring financing in a pull supply chain with a capital-constrained manufacturer selling a product through a capital-abundant retailer. Early payment is the prepayment of a wholesale cost to the manufacturer, whereas in-house factoring is a loan service provided to the manufacturer by a branch financing firm of the same retailer. We find that the retailer prefers early payment financing to bank financing when the manufacturer's production cost is low. If the retailer instead offers positive-interest in-house factoring financing to the manufacturer, then the financing equilibrium domain enlarges as compared to bank financing. Interestingly, early payment financing can outplay positive-interest in-house factoring financing if the production cost is considerably low; otherwise, vice versa. When the production cost is big enough, the retailer will not provide either early payment or in-house factoring. Furthermore, our main qualitative result sustains with an identical wholesale price across all three financing schemes and the financing equilibrium domain of early payment shrinks as demand variability grows.

Keywords: bank financing; early payment; in-house factoring; pull supply chains; newsvendor

1 Introduction

In pull supply chains, retailers place "at-once" orders in a selling season, whereas manufacturers must manage the inventory and take all the inventory risks. In the extant literature, pull supply chains are often referred to as "retailers buying from a newsvendor" and are exemplified by consignment inventory, vendor managed inventory (VMI), and drop shipping (Cachon, 2004). For example, Trek Inc., a high-end bicycles manufacturer, is willing to bear all inventory costs, while retailers

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place “at-once” orders whenever demand is realized (Cachon, 2004). Ever since Procter & Gamble, Co. and Wal-Mart Stores, Inc. began the practice of “Reengineering the Corporation,” VMI has been one of the successful business models used by many big box retailers (Çetinkaya and Lee, 2000), such as Wal-Mart, Home Depot, Amazon, and Alibaba.

Unfortunately, many manufacturers are small, capital-constrained, and lack the creditworthiness to borrow sufficient cash to fund their production. According to Chen and Gupta (2014), millions of small businesses account for 60% to 80% of all US jobs, but 43% of small business owners have been capital-constrained at least once in the last four years and could not secure any financing. As reported by the World Bank Group Enterprise Surveys (Bank, 2016), 27% of 130,000 firms across 135 countries identify “access to finance” as a major business constraint. The financial distress can further burden capital-constrained manufacturers, especially if the production lead time is long, such that the manufacturer has to produce and stock up on product before the retailer orders after the demand realizes in a short selling season (e.g., seasonal and holiday sales).

To help resolve manufacturers’ financial distress, zero-interest early payment has emerged as an alternative to traditional bank financing. In *zero-interest early payment financing* (hereafter referred to as *early payment* for brevity), a retailer will provide sufficient capital – a prepayment of the wholesale cost – for the manufacturer to carry on the contracted production. For example, big retailers, such as Costco, Amazon, Macy’s, and Walgreens, have helped fund their capital-constrained manufacturers via early payment through the C2FO financing platform (C2FO, 2016). In 2009, Wal-Mart established a “Supplier Alliance Program,” promising to pay eligible suppliers about 60 days earlier (O’Connell, 2009), and in 2015, it also extended early payment terms to about 10,000 reliable suppliers who can obtain capital in 10 days via “early real-time payment software” (Green, 2015). As reported by Marks (2015), since 2015 Home Depot has also stepped up to pay its manufacturers earlier via Taulia supply chain finance systems. JingDong, the largest online business-to-consumer retailer in China (with a $67.2 billion net revenue in 2018, according to jd.com), has also offered early credit to its manufacturers. According to Chen et al. (2016), since 2013 JingDong Finance has provided more than 30 billion RMB in advance buyer credit per year to its suppliers. While bank financing takes up about 35%-40% ($5.5-$6.4 trillion), the World Bank estimates that cash in advance represented about 19%-22% ($3-$3.5 trillion) of all trade finance arrangements in 2008 (Chauffour and Malouche, 2011). Nevertheless, to obtain a no-interest early payment, a manufacturer has few choices but to sacrifice a percentage of the early payment discount at its retailer’s discretion. For example, Wal-Mart commands a 2% discount for its early payment offer (Green, 2015).

In practice, there is another alternative buyer-financing. Instead of providing a no-interest early payment, some retailers would rather charge interest on their advance payments to manufacturers as loans via a branch financing firm. For example, JingDong uses its factoring financing branch, JingDong Finance, to offer the JingBaoBei financing service to its manufacturers as loans with a fixed interest rate (Chen et al., 2016). Amazon also provides loans to small business suppliers via its Amazon Lending program (Chen and Gupta, 2014). Different from traditional factoring financing, JingDong Finance and Amazon Lending serve as the (in-house) "third-party" financiers for their respective parent companies. Hereafter, we refer to this type of advance payment as *in-house factoring financing*.

Mathematically, we would find that early payment could be treated as a special case of in-house factoring by setting the in-house factoring interest rate to zero and ignoring the setup costs (e.g., extra employees and licensing) for a branch financing firm. To focus on the impact of a positive interest rate, we have deliberately assumed zero setup costs; however, this does not erase the factual disparity between in-house factoring financing and early payment in the following additional aspects. In *accounting*, early payment is a prepayment for the retailer's purchase from the manufacturer, whereas in-house factoring financing is a loan. In terms of *procedure*, early payment is part of the order payment and will not go through the loaning process. By contrast, in-house factoring must go through the loaning process. As a result, in-house factoring financing usually occurs when the retailer also has a financing subsidiary authorized with corresponding licenses, whereas early payment occurs as long as the retailer has sufficient capital to pay for its order upfront. In terms of *interest rate*, early payment as prepayment does not come with an interest rate, whereas in-house factoring financing typically commands a positive interest rate.

The extant literature has not documented whether early payment and in-house factoring can always outperform bank financing, although, in practice, retailers are actually selective on how to provide the financing service. The extant literature is also mute on whether it is always beneficial for the retailer to turn early payment into in-house factoring financing by charging a positive interest rate (even if the setup costs are zero) or vice versa. Therefore, it is not clear whether the retailer should offer early payment or in-house factoring, especially when bank financing is also viable to the manufacturer. If we assume retailers should offer those options, then *at what conditions can the retailer be better off offering zero-interest early payment or in-house factoring, as compared to bank financing?* Supposing the setup costs for in-house factoring branch are zero, *should the retailer always choose in-house factoring over early payment? If yes, what is the optimal interest rate? How would other factors, such as identical wholesale price and demand variability, affect the retailer's* *financing equilibrium strategy?*

1.1 Main results

To answer the above research questions, we consider a stylized pull supply chain model where a capital-constrained manufacturer sells a product through a capital-abundant retailer. The manufacturer may borrow from either the bank via *bank financing* or the retailer via no-interest *early payment or positive-interest in-house factoring* (also see Figure 1 in the Model section). We first analyze bank financing and early payment financing separately and characterize their optimal solutions respectively. We then compare early payment to in-house factoring financing.

Compared to bank financing, early payment financing demonstrates a better risk-sharing mechanism in coordinating the supply chain, because the retailer and the manufacturer each share a partial risk of the uncertainty. We find that, if the manufacturer's production cost is not too high, the retailer prefers early payment financing to bank financing.

However, the retailer has incentives to command an extremely low wholesale price under early payment financing, so both the manufacturer and the whole supply chain could be worse off under early payment compared with bank financing. The viability of bank financing can be used to leverage against early payment, so in early payment financing the retailer must enhance the wholesale price to appeal to the manufacturer. Our analysis reveals that there exists a wholesale price Pareto zone, in which both firms can be better off in using early payment financing as long as the manufacturer's production cost is sufficiently low. Even though the retailer's profit deteriorates due to the competition against bank financing, the manufacturer produces more because of their higher profit margin, which leads to a higher profit for the whole supply chain. This observation demonstrates that the competition from bank financing forces the retailer in early payment financing to surrender part of its profit to the manufacturer, so the whole supply chain can be better coordinated.

To interpret the advantage of a *no-interest* commitment in early payment, we compare it to *in-house factoring financing* with *positive* interest, a loan-type advance payment. Compared with bank financing, in-house factoring continues to exhibit the benefit of integrating finance and operations decisions by the retailer. Comparing early payment to in-house factoring, we surprisingly find that neither financing scheme can always dominate the other, and the upfront commitment of no interest in early payment can outplay positive-interest in-house factoring financing. In particular, when the production cost is low, early payment outpaces in-house factoring (the benefit increases if the latter's setup costs are not zero); however, as the production cost increases, in-house factoring financing becomes more attractive and eventually dominates early payment financing.

Although both early payment and in-house factoring are considered better risk-sharing mechanisms than bank financing, the benefits of positive interest in in-house factoring (interest benefit), a lower wholesale price (wholesale-price benefit), and a larger production quantity (production-quantity benefit) vary over the manufacturer's production cost and, thus, the retailer should adjust its interest strategy accordingly. In general, the retailer's optimal interest rate increases with the manufacturer's production cost before hitting the upper bound, which instead decreases with the production cost to warrant a reservation profit for the manufacturer when borrowing from bank financing.

Mathematically, early payment looks like a special case of the in-house factoring if we set the interest rate of in-house factoring to zero. Based on the above discussion, the optimal interest rate of in-house factoring will thus be zero when the production cost is low. However, this potential outcome is based on the assumption that the two financing procedures are the same and in-house factoring setup costs are zero, which cannot be true given that the retailer has to hire additional employees to process the factoring and to obtain financing licenses for doing so. This might explain why, in practice, we have not observed zero-interest in-house factoring.

We also extend our analysis to a scenario where the retailer would command an identical (uniform) wholesale price across all three financing schemes. Our analysis exhibits that either early payment or in-house factoring continues to dominate bank financing when the production cost is low, and the reverse is true otherwise. Provided that the production cost is low, early payment dominates other financing schemes if the wholesale price is also low; as the wholesale price increases, in-house factoring and then bank financing dominate sequentially. Like in the baseline model, the retailer can still decide whether to provide early payment or in-house factoring to manipulate its financing equilibrium choice by adjusting the identical wholesale price level. In another extension, we observe that the financing equilibrium domain of early payment shrinks as demand variability grows, because the manufacturer takes advantage of the risk-sharing mechanism to over-produce in early payment.

1.2 Related literature review

This paper is related to the literature on supply chain finance and pull supply chains. The first related research stream is on bank financing. For example, Buzacott and Zhang (2004) use a newsvendor model to investigate the interplay between inventory decisions and asset-based financing. Dada and Hu (2008) consider a capital-constrained newsvendor that can borrow from a bank and investigate conditions under which channel coordination can be achieved. Caldentey and Haugh

(2009) study a two-echelon supply chain in which the retailer is budget constrained and investigate different types of procurement contracts (wholesale contract, flexibility contract, flexibility contract with hedging) between the agents. Based on a newsvendor model, Chen et al. (2011) compare three different payment schemes and show that the payment scheme can lead to different inventory decisions. Zhou et al. (2020) explore the impact of two types of upstream firms' guarantee in bank financing on firms' performance and reveal a follower advantage in guarantor financing. Our paper also studies bank financing, but only as a benchmark case.

The second related research stream is on trade credit where a capital-abundant manufacturer can offer trade credit to a capital-constrained retailer. For example, Cai et al. (2014) examine the retailer's financing strategy when using both bank credit and trade credit under moral hazards, and use empirical data to support their theoretical results. Jing et al. (2012) show that both bank credit and trade credit can be financing equilibrium under some conditions in a model with a manufacturer selling through a capital-constrained retailer. Peura et al. (2017) prove that trade credit can soften horizontal price competition in a Bertrand competition framework. Different from the trade credit literature, the manufacturer is no longer capital-abundant but rather capital-constrained in our model.

The third related research stream is on pull systems. Recent decades have witnessed more increasingly dominant, large, centrally managed "power retailers," such as chain supermarkets, mass merchandisers, wholesale clubs, and category killers. Accordingly, numerous theories have been proposed to explain these powerful downstream retailers. For example, Cachon (2004) compares three different contracts, namely push, pull, and advance-purchase discount, to study the impact of contracts on inventory risk allocation. In the pull system, the retailer behaves as a leader and decides the wholesale price. Dong and Zhu (2007) illustrate how the inventory decision rights and ownership are shifted and/or shared between a supplier and a retailer under a two-wholesale-price contract, resulting in push, pull, or advance-purchase discount contracts. Yang et al. (2018) explore the impact of risk averse attitude on push and pull contracts and show that a push contract can outperform a pull contract for the whole supply chain when the supplier is sufficiently more risk averse than the retailer. Our model is similar to these pull supply chains; however, all these papers ignore financing decisions.

The fourth, and most related, research stream is on financing the suppliers. For example, Tunca and Zhu (2018) build a theoretical model to compare traditional bank financing and buyer intermediation in supplier financing (BIF), and they demonstrate that BIF induces lower wholesale prices and higher order quantities. Several papers have also been built upon pull supply chains with buyer financing. Deng et al. (2018) explore an assembly system with one assembler and multiple heterogeneous suppliers. They find that the assembler can still benefit from offering buyer financing even if its capital opportunity cost is higher than the bank's risk-free interest rate. Reindorp et al. (2015) study the equilibrium of purchasing ordering financing in the case of "buying from the newsvendor." However, differently from those studies, we investigate early payment financing with zero interest rate and examine the impact of the manufacturer's production cost on the retailer's financing equilibrium, which is not the focal point of those previous models.

The work by Tang et al. (2018) is closely related to our article and worth special mentioning. Tang et al. (2018) capture the interactions among a small supplier (Stackelberg follower), a manufacturer (Stackelberg leader) and a bank, to analyze how information influences the relative efficiency of two financing schemes: purchasing order financing (POF) and buyer direct financing (BDF). Theoretically, their POF is similar to our bank financing and their BDF is similar to our in-house factoring financing. But, their model and ours are different in multiple ways. First, for tractability, their model assumes that "the demand faced by the manufacturer is assumed to be known and is normalized to 1 without loss of generality." Differently, we assume that demand is random following a general probability distribution function, which is typical in the newsvendor literature. Second, whereas their model focuses on the supply risk and effort information asymmetry, our model focuses on the value of early payment and gains insights on when to choose which financing scheme under demand uncertainty. Third, Tang et al. (2018) "find that when the manufacturer and the bank have symmetric information, POF and BDF yield the same payoffs for all parties irrespective of the manufacturer's control advantage under BDF." They suggest that POF and BDF can outperform each other only under asymmetric information. Different from their finding, our analysis suggests that, even under symmetric information, early payment/in-house factoring or bank financing can outperform each other under demand uncertainty, depending on the production cost level. Fourth, Tang et al. (2018) mainly compare POF to BDF, whereas we first compare early payment to bank financing, then compare in-house factoring to bank financing, and finally compare in-house factoring to early payment.

2 The Model

We consider a pull supply chain with one capital-constrained upstream firm (hereafter referred to as manufacturer), and one capital-abundant downstream firm (hereafter referred to as retailer), where the manufacturer produces and sells a product via the retailer to the market. The manufacturer charges a procurement (wholesale) price w per unit, whereas the retailer sells the product at a

Table 1: Notations and abbreviations

p	retail price
c	unit production cost for the manufacturer
v	unit product salvage value
i	$= b, e, I$, denotes bank financing, early payment, and in-house factoring financing, respectively
w_i	wholesale price at $i = b, e, I$, and w_i^* is the equilibrium
D	random variable representing uncertain market demand
	$f(\cdot)$ and $F(\cdot)$ are density and cumulative probability function of D, respectively
$\bar{F}(Q)$	$= 1 - F(Q)$
$h(Q)$	$= f(Q)/\bar{F}(Q)$
$H(Q)$	$= Qh(Q)$
$(x)^+$	$= \max[x, 0]$
Q	production quantity
$S(Q)$	$= \mathbb{E}\min[D, Q]$
$J(Q)$	$= 1 + h(Q)S(Q)/\bar{F}(Q)$
$V(Q)$	$= Q\bar{F}(Q)$
$L_e(Q)$	$= (c - v)Q_e/(w_e - v)$
$L_I(Q)$	$= ((c(1 + r_I) - v)Q_I/(w_I - v)$
$Y(Q)$	$= \int_0^Q DdF(D)$
$\tilde{Q}$	satisfies $H(\tilde{Q}) = 1$
Q^0	satisfies $(p - v)\bar{F}(Q^0) = c - v$
$\Gamma^0(c)$	$= (p-v)S(Q^0) - (c - v)Q^0$
r_f	risk-free interest rate and $r_f = 0$ in this paper
r_i	interest rate for financing option i, and $i = b, I$. r_i^* is the optimal interest rate at $i = b, I$
$\hat{r}_I$	solving $\Pi_I(Q_I^*, w_I = p, \hat{r}_I) = \Pi_b^*$
$\hat{r}_I^M$	solving $\Omega_I^M(Q_I^*, w_I^M(\hat{r}_I^M)) = \Omega_I(Q_I^*, w_I^M(0))$
Q_i	production quantity in financing option i
$Q_i^*(w_i)$	optimal production quantity for a given wholesale price w_i in i
$\Pi_i(Q_i, w_i)$	profit of the manufacturer in $i = b, e, I$, and is Π_i^* in equilibrium
$\Omega_i(Q_i, w_i)$	profit of the retailer in $i = b, e, I$ model, and is Ω_i^* in equilibrium
$\Gamma_i(Q_i, w_i)$	profits of total supply chain in $i = b, e, I$ model, and is Γ_i^* in equilibrium
$\bar{c}_i$	$\bar{c}_e = \frac{(p-v)\bar{F}(\tilde{Q})}{1+2\bar{F}(\tilde{Q})} + v$ and $\bar{c}_b = (p - v)\bar{F}(\tilde{Q})/J(\tilde{Q}) + v$
c^0	$= (p - v)\bar{F}(\tilde{Q}) + v$, and $c^0 \in (\max\{\bar{c}_e, \bar{c}_b\}, \check{c})$
$\check{c}$	$= (p - v)S(\tilde{Q})/\tilde{Q} + v$
$\dot{c}_i$	satisfies $\Gamma_i(\tilde{Q}) = \Omega_b^*(c)$, $i = e, I$ and $\dot{c}_i \in (\max\{\bar{c}_e, \bar{c}_b\}, \check{c})$
$w_i^M(c)$	satisfies $\Pi_i(Q_i^*, w_i^M(c)) = \Pi_b^*$, and $i = e, I$
$w_i^R(c)$	satisfies $\Omega_i(Q_i^*, w_i^R(c)) = \Omega_b^*$, and $w_i^R(c) \in (w_i^*, p)$, $i = e, I$
$\hat{c}_i$	satisfies $w_i^M(\hat{c}_i) = w_i^R(\hat{c}_i)$ and $\hat{c}_i \in (v, \dot{c}_i)$, $i = e, I$
$\ddot{c}_e$	satisfies $\Gamma_e^*(\ddot{c}_e) = \Gamma_b^*(\ddot{c}_e)$, and belongs in $(\max\{\bar{c}_e, \bar{c}_b\}, \dot{c}_e)$
$\tilde{c}_1$	solving $(w_I^M(\hat{r}_I) - v)\bar{F}(Q^0(c)) = (c(1 + \hat{r}_I) - v)\bar{F}(L_I(Q^0(c)))$
$\tilde{c}_2$	solving $\Omega_I(Q_I^*, w_I^M(\hat{r}_I)) = \Omega_I(Q_I^*, w_I^M(r_I = 0)))$
$\tilde{c}_3$	solving $(w_I^M - v)\bar{F}(Q^0(c)) = (c - v)\bar{F}(L_e(Q^0(c)))$

retail price p. Consistent with most newsvendor models (see, e.g., Cachon, 2004), p is assumed to be exogenous for tractability. The unit production cost for the manufacturer is c, and the unit product salvage value is v, where $v < c$, so the manufacturer has no incentive to produce unlimited amount of products. To avoid triviality, we assume $p \geq w \geq c$.

Conforming to the newsvendor literature, we consider a one-period, pull, newsvendor model, like in Cachon (2004), Chen and Gupta (2014), Dong and Zhu (2007), Ge and Qiu (2007), Wang et al. (2014), and Yang et al. (2018), and assume that the retailer is the leader and decides the wholesale price at first, and subsequently the manufacturer (follower) decides the production quantity. This assumption reflects a situation where the retailer has a short selling season (e.g., seasonal or holiday sales) while the manufacturer has a long production lead time and there is only one production opportunity, such that the manufacturer has to produce and stock up on product well before the retailer's at-once order is placed after the demand is realized. Both the retailer and the manufacturer are risk neutral and attempt to maximize their profits.

We assume demand, D, is random following a probability distribution function of $f(D)$ and cumulative distribution function of $F(D)$. The demand distribution has the Increasing Failure Rate (IFR) property with a failure rate $h(D) = f(D)/\bar{F}(D)$, where $\bar{F}(D) = 1 - F(D)$. The failure rate of the demand distribution $h(D)$ is assumed to be both increasing with D and convex on its support. Let Q be the manufacturer's production quantity and $H(Q) = Qh(Q)$. Let $\tilde{Q}$ be the threshold value satisfying $H(\tilde{Q}) = 1$. For tractability, we assume the manufacturer has zero initial working capital and must rely on outside sources to finance its operations by covering the producing cost, cQ. The manufacturer has limited liability, which is in line with Jing et al. (2012).

There are three alternatives to finance the manufacturer's operations. In the first financing scheme, *bank financing*, the manufacturer borrows funds from a third party financial institution (e.g., a bank). Following the convention in bank financing literature, we assume the bank resides in a competitive financing market with a risk-free interest rate, r_f, which is normalized to zero for simplicity (see, e.g., Dotan and Ravid, 1985; Brennan et al., 1988; Dammon and Senbet, 1988; Jing et al., 2012).

In the second financing scheme, *zero-interest early payment financing (*also referred to as *early payment* for brevity*)*, the manufacturer collects *partial* payment (*prepayment* in accounting) from the retailer in advance to cover the production cost, cQ. Early payment has helped many small or startup manufacturers, who lack creditworthiness or credit history, resulting no bank providing financing to them, which is not uncommon in developing economies (Bank, 2016). In practice, early payment is not a loan, because there is no interest rate imposed on the payment, and the prepayment is only part of the wholesale revenue to be paid to the manufacturer. The manufacturer collects the remaining payment upon delivery of the product to the retailer. However, if the demand is too low such that the retailer's revenue cannot cover the early payment, the retailer will absorb the loss provided that the manufacturer's production has already occurred. Given that the production cost equals the amount of early payment, in the case of insufficient demand, the manufacturer receives no additional payment and its profit is zero.

We further discuss another buyer-financing mechanism, *in-house factoring financing* (in Section 4.1), in which the retailer provides a loan to the manufacturer with a positive interest rate of r_I. The loan size is cQ to warrant the production. Mathematically, early payment looks like a special case of the in-house factoring financing; however, different from early payment financing, the interest rate is usually positive in in-house factoring financing (see, e.g., de Booth et al., 2015; Chen et al., 2016). To the best of our knowledge, we have not observed zero-interest in-house factoring in practice. More importantly, early payment differs from in-house factoring financing in practical procedure. In early payment, the portion of cash transferred from the retailer to the manufacturer is a prepayment for an order that the retailer will purchase from the manufacturer. Therefore, the early payment is part of the total order payment, rather than a loan, and it will not go through the loaning process. By contrast, in-house factoring financing is a buyer-financing loan and must go through the loaning process. Meanwhile, in-house factoring financing typically occurs when the retailer also has a financing subsidiary, whereas early payment does not require a financing subsidiary so long as the retailer has sufficient capital to pay for its order upfront. In practice, in-house factoring also requires factoring specialists and financing licenses. To focus on the interest disparity between early payment and in-house factoring, we assume setup costs and license fees are zero and only use them as a tie breaker between early payment and in-house factoring.

We use subscript $i = b,\ e,\ I$ to denote bank financing, early payment financing, and in-house factoring financing, respectively. For instance, w_b denotes the wholesale price under bank financing. We let Π, Ω, and Γ represent the expected profits of the manufacturer, the retailer, and the whole supply chain, respectively.

We use Figure 1 to depict the event sequence in these financing schemes. In the beginning of the time period, the retailer offers a purchase order and decides whether to offer zero-interest early payment or in-house factoring financing (with a positive interest rate). If the retailer does not offer early payment or in-house factoring financing, then the manufacturer has to choose bank financing. Accordingly, as the leader, the retailer provides a purchase price w_i for each financing scheme i, the manufacturer determines the production quantity Q_i, and then the bank offers an interest rate $r_b(Q_b, w_b)$ in bank financing based on the fairly priced rule. In the end of the time period, the demand is realized and the retailer collects the revenue and pays the manufacturer the rest of the payment, $w_i S(Q_i)$, minus the loan or early payment where $S(Q_i) = \mathbb{E}\min[D, Q_i]$. If the product is overstock, the manufacturer also receives $v\mathbb{E}(Q_i - D)^+$ salvage income. In bank financing, the manufacturer repays the bank. Overall, the total investment revenue for the lender in financing scheme i is $\mathbb{E}\min[w_i \min[Q_i, D] + v(Q_i - D)^+, c(1 + r_i)Q_i]$, where there is no interest for early payment (i.e., r_e is zero).

The subgame perfect equilibrium of the game is solved backwards. We assume a tie-breaking rule that the retailer will choose early payment over in-house factoring financing if it is indifferent between the two because of the aforementioned setup costs.

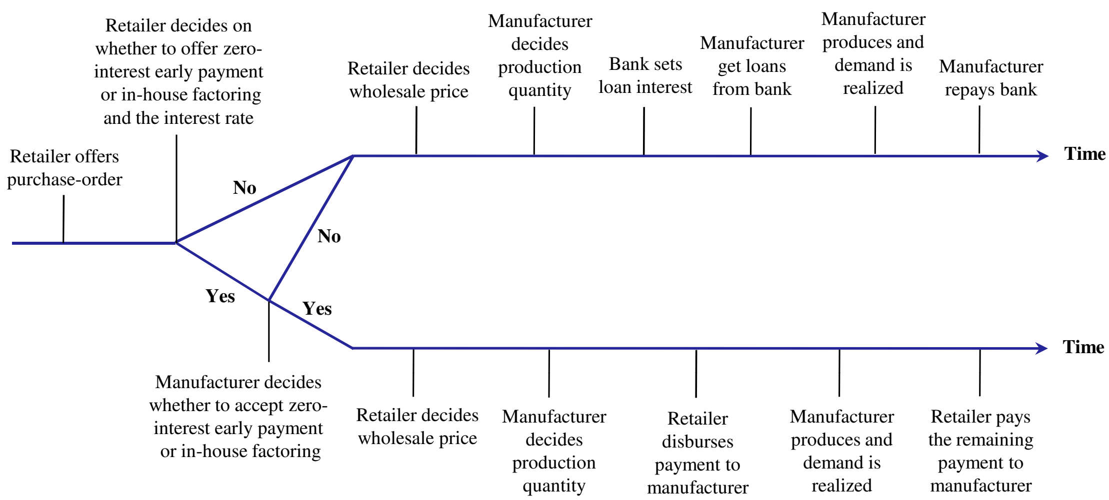


Figure 1: Sequences of events in different financing schemes.

We first focus on comparing early payment to bank financing in Section 3 and then study in-house factoring in Section 4. To highlight the robustness of our qualitative outcomes, we also consider the case in which the retailer commands a uniform/identical wholesale price among different financing schemes in Section 5.1. We also consider a centralized supply chain without capital constraint as a benchmark. When $Q = Q^0$, the total supply chain profit $\Gamma(Q) = (p-v)S(Q) - (c-v)Q$ reaches the maximum, where Q^0 satisfies $(p - v)\bar{F}(Q^0) = c - v$. All notations are listed in Table 1.

3 Analysis of Zero-Interest Early Payment

Provided that there is no interest in early payment, one concern is whether early payment is beneficial to the retailer. To address this concern, we use this section to focus on early payment as prepayment and compare it to bank financing. If the retailer offers early payment to the manufacturer, then the manufacturer would choose either early payment or bank financing. In this section, we start with the isolated cases of bank financing and early payment financing and then compare them.

3.1 Bank financing

If the retailer does not offer early payment or the manufacturer refuses to accept early payment, the manufacturer will manage to borrow bank credit. In bank financing, the retailer first chooses a wholesale price w_b and then the manufacturer chooses the production quantity Q_b. The capital-constrained manufacturer then borrows a loan, cQ_b, from the bank to produce Q_b units of product.

Since the bank market is perfectly competitive with $r_f = 0$, the bank makes zero expected profit by lending to the retailer. For any Q_b (or equivalently, loan size cQ_b) chosen by the manufacturer, the interest rate r_b^* equates the bank's expected return to its lending costs. Similar to the vast literature on bank financing (see, e.g., Jing et al., 2012), the bank determines the interest rate based on the following fairly priced equation:

$$\mathbb{E}\min[w_b \min[D, Q_b]+v(Q_b - D)^+, c(1+r_b^*)Q_b] = cQ_b. \tag{1}$$

The manufacturer's optimal production problem can be formulated as:

$$\max_{Q_b} \Pi_b(Q_b) = \mathbb{E}\left[w_b \min[D, Q_b]+v(Q_b - D)^+ - c(1+r_b^*)Q_b\right]^+. \tag{2}$$

Substituting Eq. (1) into Eq. (2), we obtain the manufacturer's profit as follows:

$$\begin{aligned}\Pi_b(Q_b) &= \mathbb{E}\left[w_b \min[D, Q_b] + v(Q_b - D)^+\right] - \mathbb{E}\min[w_b \min[D, Q_b] + v(Q_b - D)^+, c(1+r_b^*)Q_b] \\ &= (w_b - v)S(Q_b) - (c - v)Q_b.\end{aligned}$$

The retailer's profit function can be written as $\Omega_b(w_b) = (p - w_b)S(Q_b)$. Solving the first order condition for $\Pi_b(Q_b)$, we obtain the manufacturer's optimal production level Q_b^* to satisfy $\bar{F}(Q_b^*) = (c - v)/(w_b - v)$. Based on the one-to-one mapping between Q_b^* and w_b, we have $w_b = (c-v)/\bar{F}(Q_b^*)+v$. The following result describes the firms' equilibrium strategies under bank financing.

Lemma 1 *Consider bank financing.*

1. *The manufacturer's optimal production quantity Q_b^* satisfies $(p - v)\bar{F}(Q_b^*) = (c - v)J(Q_b^*)$;*
2. *The retailer's optimal wholesale price is $w_b^* = (c - v)/\bar{F}(Q_b^*) + v$;*

3. *The bank's optimal interest rate* r_b^* *satisfies* $S\left(\frac{(1+r_b^*)c-v}{c-v}V(Q_b^*)\right) = V(Q_b^*)$.

Because the bank earns zero expected profit (i.e., $r_f = 0$), the manufacturer's expected cost of using bank financing is identical to that of using its own capital should it have enough capital. From the retailer's perspective, with the bank's help, a capital-constrained manufacturer behaves like one with sufficient capital, which is consistent with the extant literature that the competitive bank market separates the manufacturer's finance decisions from its operations decisions.

While bank financing helps both the manufacturer and the retailer, such benefits are affected by the manufacturer's production cost.

Corollary 1 *Under bank financing, the retailer's profit and the whole supply chain's profit (i.e.,* $\Omega_b^*(c)$ *and* $\Gamma_b^*(c)$*, respectively) decrease with the production cost (i.e.,* $c \in (v, p]$*).*

Because the wholesale price increases as the production cost rises, the retailer's marginal profit decreases. The manufacturer's marginal profit also decreases, even though the wholesale price climbs. Consequently, the manufacturer's production quantity declines and profits decrease for the retailer and the whole supply chain.

3.2 Zero-interest early payment

In early payment, the retailer first sets the procurement price w_e, the manufacturer then decides to produce Q_e, and the retailer disburses an early payment, cQ_e, as a prepayment to the manufacturer. After the demand is realized, the retailer receives a revenue of $p\min[D, Q_e]$ and pays the remaining payment $(w_e \min[D, Q_e] + v(Q_e - D)^+ - cQ_e)^+$ to the manufacturer. For simplicity, we use the following notations, $L_e(Q_e) = (c-v)Q_e/(w_e - v)$ and $Y(Q_e) = \int_0^{Q_e} D dF(D)$. Then, we can rewrite the manufacturer's profit function under zero-interest early payment financing as:

$$\begin{aligned} \Pi_e(Q_e, w_e) &= \mathbb{E}\left[\max[w_e \min[D, Q_e] + v(Q_e - D)^+, cQ_e] - cQ_e\right] \\ &= (w_e - v)\left[Y(Q_e) - Y(L_e(Q_e)) + V(Q_e) - V(L_e(Q_e))\right]. \end{aligned} \quad (3)$$

Solving for the manufacturer's optimal production quantity, we have the following outcome.

Lemma 2 *Consider zero-interest early payment financing with any given* w_e*.*

1. *The manufacturer's optimal production quantity* Q_e^* *is solved by* $(w_e - v)\bar{F}(Q_e^*) = (c - v)\bar{F}(L_e(Q_e^*))$*, if* $w_e > c$*; and* $Q_e^* = \tilde{Q}$ *if* $w_e = c$*;*
2. Q_e^* *increases with* w_e*; and* $Q_e^* \geq \tilde{Q}$*.*

Lemma 2 indicates that the manufacturer can obtain a unique optimal production quantity for any given w_e. Substituting Q_e^* into Eq. (3), we can rewrite the manufacturer's payoff function as:

$$\Pi_e(Q_e^*, w_e) = (w_e - v)(Y(Q_e^*) - Y(L_e(Q_e^*))). \tag{4}$$

The retailer's expected profit can then be rewritten as:

$$\begin{aligned}
\Omega_e(Q_e^*, w_e) &= (p-v)S(Q_e^*) - (c-v)Q_e^* - \mathbb{E}[(w_e - v)\min[D, Q_e^*] - (c-v)Q_e^*]^+ \\
&= (p-w_e)S(Q_e^*) - (c-v)Q_e^* + \mathbb{E}\min[(w_e - v)\min[D, Q_e^*], (c-v)Q_e^*] \\
&= (p-w_e)S(Q_e^*) + (w_e - v)Y(L_e(Q_e^*)) - (c-v)Q_e^* F(L_e(Q_e^*)).
\end{aligned} \tag{5}$$

Based on Eq. (4) and (5), we obtain the whole supply chain's profit $\Gamma_e(Q_e^*, w_e) = (p-v)S(Q_e^*) - (c-v)Q_e^*$ and the following properties.

Lemma 3 *Consider zero-interest early payment financing.*

1. $\Pi_e(Q_e^*, w_e)$ *increases with* w_e*;*
2. $\Omega_e(Q_e^*, w_e)$ *is a unimodal function of* w_e*;*
3. $\Gamma_e(Q_e^*, w_e)$ *is a unimodal function of* w_e*.*

The manufacturer's profit straightforwardly increases with the wholesale price because of the higher profit margin. Lemma 3 also shows that the retailer's profit is a unimodal function of w_e. This result occurs because the retailer can benefit from a larger production quantity as the wholesale price grows, when w_e is sufficiently small. But, the benefit shrinks as the wholesale price substantially increases and overshadows the benefit of a larger production quantity. As the leader, the retailer can thus identify the unique optimal wholesale price for its procurement from the manufacturer.

The retailer's optimal wholesale price and the manufacturer's optimal production quantity can be further characterized by the following theorem.

Theorem 1 *Consider zero-interest early payment financing.*

1. *There exists a production cost threshold point* $\bar{c}_e = \frac{(p-v)\bar{F}(\tilde{Q})}{1+2\bar{F}(\tilde{Q})} + v$*, such that, if* $v < c \leq \bar{c}_e$*, the retailer's optimal wholesale price* w_e^* *satisfies* $d\Omega_e(Q_e^*, w_e)/dw_e = 0$*; otherwise if* $\bar{c}_e < c \leq p$*, then* $w_e^* = c$*.*
2. *The optimal order quantity* Q_e^* *and the retailer's optimal profit* Ω_e^* *(weakly) decrease with* c*.*

Theorem 1 reveals that when the manufacturer's production cost is substantially higher (i.e., $\bar{c}_e < c \leq p$), we have $w_e^* = c$, so the retailer squeezes all surplus from the manufacturer. This result occurs because, provided that the production cost is substantially high, the retailer's profit margin is razor-thin and it has to bear all financial risk if the manufacturer defaults (caused by low demand). In this situation, the manufacturer's expected payoff is zero (i.e., $\Pi_e^* = 0$), and the retailer obtains all supply chain profit (i.e., $\Omega_e^* = \Gamma(\tilde{Q}) = (p-v)S(\tilde{Q}) - (c-v)\tilde{Q}$). If the production cost is low (i.e., $v < c \leq \bar{c}_e$), then the retailer grants a profit margin to the manufacturer. As c increases, the retailer's optimal profit Ω_e^* decreases with c. In fact, if the manufacturer's production cost surmounts $\bar{c}_e$ and reaches a higher threshold point $\check{c} = (p-v)S(\tilde{Q})/\tilde{Q}+v$, then the retailer also earns zero profit. If $c \in (\check{c}, p]$, then $\Omega_e^* \leq 0$. Obviously, the retailer has no incentive to offer early payment if the manufacturer's production cost is too high (i.e., $c > \check{c}$), because when the production cost is high, the retailer commands a wholesale price equal to the production cost such that the manufacturer earns zero profit. The manufacturer thus becomes more risk-seeking by maintaining a relatively high production level to earn a positive profit if demand is high or default otherwise, which hurts the retailer.

Corollary 2 *There exists a unique point $c^0 = (p-v)\bar{F}(\tilde{Q})+v$, such that, when $c = c^0$, the zero-interest early payment coordinates the supply chain (i.e., $\Gamma^0(c^0) = \Gamma_e^*(c^0)$).*

Corollary 2 implies that *the manufacturer's risk-taking in production can lead to the first-best outcome (a perfect coordination effect) in early payment, which does not occur in bank financing.* Overall, the early payment's coordination effect is better when the production cost is neither too high nor too low. When the production cost is low, the manufacturer keeps a substantial portion of surplus such that the double marginalization negatively impacts the whole supply chain profit. When the production cost is high, as the preceding discussion suggests, the financial risk cost to the retailer is overwhelming, such that it hurts the retailer and the whole supply chain. Nevertheless, the above coordination effect also reveals the risk-sharing mechanism embedded in early payment when compared to bank financing, which is discussed more in the next section.

3.3 Comparison of early payment to bank financing

We now compare the firms' performances in the above two financing schemes. As shown in Corollary 1 and Theorem 1, the retailer's optimal profits in both bank financing and early payment decrease with the production cost. In bank financing, we can identify a threshold point $\bar{c}_b = (p-v)\bar{F}(\tilde{Q})/J(\tilde{Q})+v$, where the manufacturer's optimal production quantity $Q_b^* = \tilde{Q}$ when $c = \bar{c}_b$. If $c < \bar{c}_b$, we have $Q_b^* > \tilde{Q}$; otherwise $Q_b^* \leq \tilde{Q}$. In early payment, as Theorem 1 describes, if $c = \bar{c}_e$, then $Q_e^* = \tilde{Q}$. Comparing the retailer's profits in bank credit and early payment leads to the following outcome.

Theorem 2 *1. For any $c \in (v, p]$, $w_e^* \leq w_b^*$ and $\Pi_b^* > \Pi_e^*$;*

2. *There exists a unique threshold point* $\dot{c}_e \in (\max\{\bar{c}_e, \bar{c}_b\}, \check{c})$ *where* $\dot{c}_e$ *satisfies* $\Gamma(\tilde{Q}) = \Omega_b^*(\dot{c}_e)$, *such that, for the retailer, early payment financing outperforms bank financing (i.e.,* $\Omega_e^* \geq \Omega_b^*$*) if* $v < c \leq \dot{c}_e$*; otherwise,* $\Omega_e^* < \Omega_b^*$*.*

Compared with bank credit, early payment brings about several interactive effects from the retailer's perspective. On the one hand, financing the manufacturer shifts partial demand uncertainty risk from the manufacturer to the retailer (*financial-risk effect*). On the other hand, this cost-sharing mechanism incentivizes a higher production quantity from the manufacturer (*production-quantity effect*). Early payment also commands a discount for the retailer's procurement cost (*wholesale-discount effect, i.e.,* $w_e^* \leq w_b^*$). When the production cost is lower, the manufacturer's default risk can be better contained. So, the production quantity effect and wholesale discount effect in early payment financing stands out (see Figure 2 when $c < \dot{c}_e$ =0.626). However, when the production cost is substantially high (i.e., $\dot{c}_e \leq c \leq p$), the retailer has little room to further push down the wholesale price while also facing a higher financial risk. Thus, the negative financial risk effect overtakes the production quantity effect and wholesale discount effect. Therefore, for the retailer, early payment falls behind.

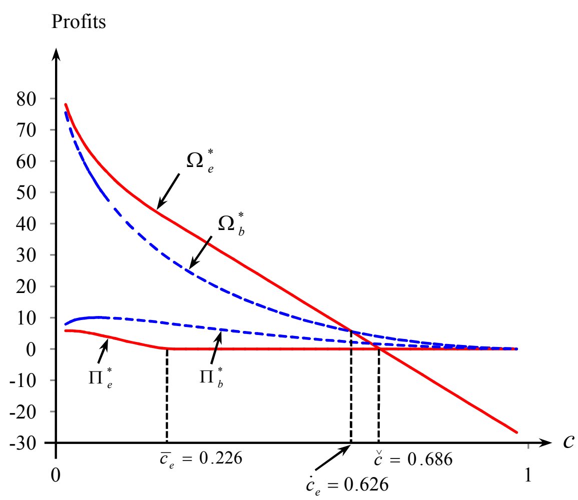


Figure 2: Firms' profit comparison between early payment and bank financing.

The retailer's gain is the manufacturer's loss. As Figure 2 depicts, the manufacturer always prefers bank financing in the domain (i.e., $\Pi_b^* \geq \Pi_e^*$), where the retailer prefers early payment. Therefore, as long as the manufacturer can access bank financing, early payment is not sustainable and the retailer will never achieve its optimal profit unless the retailer compromises some profits to the manufacturer. We next explore such a compromise in financing equilibrium.

For consistency, in all graphs, we assume that $p = 1$, demand follows a Gamma distribution with $\mu = 100$, and $\sigma/\mu = 0.9$ throughout the paper unless mentioned otherwise.

3.4 Early payment Pareto zone and financing equilibrium domain

Given that both early payment and bank credit are viable, for the manufacturer to choose early payment over bank credit, the former must generate at least the same amount of profit as the latter. Accordingly, the retailer must sufficiently raise the wholesale price in early payment to make it attractive to the manufacturer. We use the following lemma to identify this wholesale price threshold.

Lemma 4 *Consider from the manufacturer's perspective. For any $c \in (v, p]$,*

1. *there exists a unique wholesale price threshold point $w_e^M(c)$ such that the manufacturer earns the same expected profit under early payment financing as that under bank financing (i.e., $\Pi_e(Q_e^*, w_e^M(c)) = \Pi_b^*$);*
2. *$w_e^*(c) < w_e^M(c) \leq w_b^*(c)$.*

Lemma 4 demonstrates that the manufacturer is willing to adopt early payment financing as long as the retailer elevates the wholesale price to the level of $w_e^M(c)$. Because there is a positive interest rate in bank financing, the threshold wholesale price $w_e^M(c)$ must be lower than the optimal wholesale price in bank financing but higher than that in early payment (i.e., $w_e^*(c) < w_e^M(c) \leq w_b^*(c)$).

Similarly, we can identify another threshold wholesale price from the retailer's perspective.

Lemma 5 *From the retailer's perspective, if $v < c < \dot{c}_e$, there exists a unique threshold point $w_e^R(c) \in (w_e^*, p)$ such that the retailer earns the same profit under early payment as that under bank financing (i.e., $\Omega_e(Q_e^*, w_e^R(c)) = \Omega_b^*$).*

Lemma 5 reveals another threshold wholesale price at which the retailer can earn the same profit in both financing schemes if the production cost is not too high (i.e., $v < c < \dot{c}_e$). Recall from Theorem 2 that when $c = \dot{c}_e$, the retailer earns the same profit in both financing schemes. So, in early payment, if $v < c < \dot{c}_e$, the retailer can sacrifice some profit to the manufacturer by increasing the wholesale price up to $w_e^R(c)$.

As displayed in Lemma 4 and Lemma 5, both threshold wholesale prices are higher than the retailer's ideal optimal wholesale price. A concern then arises: Can the manufacturer and the retailer find a common ground so early payment emerges as the financing equilibrium? Our results say Yes.

Theorem 3 *1. There exists a critical point $c = \hat{c}_e \in (v, \dot{c}_e)$, where $w_e^M(\hat{c}_e) = w_e^R(\hat{c}_e)$, such that both the retailer and the manufacturer are indifferent between early payment and bank financing;*

2. [Pareto zone] For any $c \leq \hat{c}_e$, there exists a Pareto zone for wholesale pricing in early payment, that is $w_e \in [w_e^M, w_e^R]$, such that both the retailer and the manufacturer prefer early payment to bank financing (i.e., $\Pi_e(Q_e^*, w_e) \geq \Pi_b^*$ and $\Omega_e(Q_e^*, w_e) \geq \Omega_b^*$);

3. [Financing equilibrium] When $v < c < \hat{c}_e$, the unique sub-game perfect financing equilibrium is early payment financing; when $\hat{c}_e \leq c \leq p$, the financing equilibrium is bank financing.

Theorem 3 first confirms that the retailer and the manufacturer can find a common ground. At the threshold $\hat{c}_e$, the two threshold wholesale prices (i.e., $w_e^M(\hat{c}_e)$ and $w_e^R(\hat{c}_e)$) intersect, as depicted in Figure 3, so the retailer and the manufacturer are indifferent to both financing schemes. As the production cost decreases, the retailer has a larger profit margin to share with the manufacturer. As illustrated in Figure 3, *both firms can benefit from early payment above bank financing for any wholesale price in the Pareto zone.*

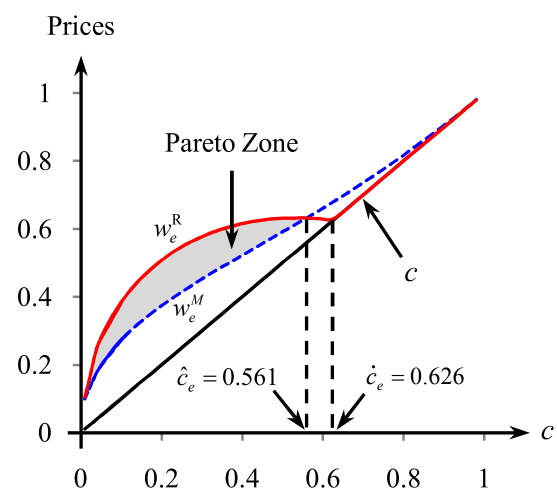


Figure 3: Pareto zone for wholesale pricing in early payment.

Given that the manufacturer would choose early payment as long as it is as profitable as bank financing, the retailer has incentives to set the wholesale price as low as possible to ultimately hit the manufacturer's indifference point (i.e., $w_e^M(c)$). However, due to the wholesale price increase (i.e., $w_e^* < w_e^M(c)$, see Lemma 4), the positive wholesale discount effect shrinks. Nevertheless, because the manufacturer is better motivated to produce more products, the positive production quantity effect rises, which pushes up the negative financial-risk effect. As long as the production cost is sufficiently low (i.e., $v < c < \hat{c}_e$), early payment can still outperform bank credit.

Because the retailer has to surrender a profit margin to the manufacturer, the financing equilibrium domain (i.e., $v < c < \hat{c}_e$) is smaller than the early payment profitable domain (i.e., $v < c \leq \dot{c}_e$ in Theorem 2) given $\dot{c}_e > \hat{c}_e$. However, because $w_e^* < w_e^M(c)$, the manufacturer produces more products in financing equilibrium than in the isolated early payment. As a result, the total supply chain profit is greater in financing equilibrium than in the isolated early payment scenario. This observation thus exposes the detriment of letting the leader control both operations and financing decisions. Without competition, the leader has incentives to jointly maneuver the finance and operations decisions for its own benefit, which significantly discourages the follower from fully contributing to the supply chain (e.g., the manufacturer produces less); accordingly, the whole supply chain suffers. Therefore, *the competition from an outside option (bank financing) suppresses the retailer's greed and brings out a higher supply chain welfare in early payment.*

4 In-House Factoring and Comparison

One conspicuous feature of early payment is that the manufacturer pays no interest for early payment, whereas the retailer mandates a lower wholesale price in return. One question naturally arises: Why doesn't the retailer charge positive interest on advance payments upfront and convert it into a loan-like in-house factoring? This section is devoted to answering this question by characterizing in-house factoring and comparing it to bank financing and early payment.

4.1 Positive-interest in-house factoring

To focus on the factoring effect, we deliberately assume that a retailer charges an exogenous positive interest rate r_I on its advance payment to the manufacturer, which is consistent with the practice. For example, in the practice of JingDong Finance, the interest rate has been stable around 9% (see, e.g., Chen et al., 2016). According to de Booth et al. (2015), Netherlands's factor interest rate has been relatively stable at about 1.5%. Theoretically, an optimal endogenous interest rate can be attained as discussed in Theorem 6 in Section 4.3, but this is not assumed in most of our discussion to serve our aforementioned focus.

For any given positive interest rate r_I, the retailer first sets the procurement price w_I, the manufacturer then decides to produce Q_I, and the retailer disburses an in-house factoring financing loan, cQ_I, to the manufacturer. After the demand is realized, the retailer receives a revenue of $p\min[D, Q_I]$ and pays the remaining payment $(w_I \min[D, Q_I] - cQ_I(1 + r_I))^+$ to the manufacturer. We denote $L_I(Q_I) = (c(1 + r_I) - v)Q_I/(w_I - v)$ and $Y(Q_I) = \int_0^{Q_I} DdF(D)$. Under in-house factoring financing, for any given r_I, the manufacturer's profit function can then be rewritten as follows:

$$\begin{aligned}\Pi_I(Q_I, w_I) &= \mathbb{E}\left[w_I \min[D, Q_I] + v(Q_I - D)^+ - c(1 + r_I)Q_I\right]^+ \\ &= (w_I - v)\left[Y(Q_I) - Y(L_I(Q_I)) + V(Q_I) - V(L_I(Q_I))\right]. \end{aligned} \tag{6}$$

Let Q_I^* be the manufacturer's optimal solution in Equation (6). Solving this optimal problem, we obtain the manufacturer's production quantity Q_I^* satisfying $(w_I - v)\bar{F}(Q_I^*) = (c(1 + r_I) - v)\bar{F}(L_I(Q_I^*))$ if $w_I > c(1 + r_I)$, and $Q_I^* = \tilde{Q}$ if $w_I = c(1 + r_I)$. Similarly, we can further deduce that Q_I^* increases with w_I, and $Q_I^* \geq \tilde{Q}$.

We submit Q_I^* into the retailer's profit function and obtain,

$$\begin{aligned}\Omega_I(Q_I^*, w_I) &= (p - w_I)S(Q_I^*) + \mathbb{E}\min[(w_I - v)\min[D, Q_I^*], (c(1 + r_I) - v)Q_I^*] - (c - v)Q_I^* \qquad (7) \\ &= (p - w_I)S(Q_I^*) + (w_I - v)Y(L_I(Q_I^*)) + (c(1 + r_I) - v)Q_I^*\bar{F}(L_I(Q_I^*)) - (c - v)Q_I^* \qquad (8)\end{aligned}$$

Solving the retailer's problem, we can obtain similar results to those in Lemma 3 that, under in-house factoring financing, $\Pi_I(Q_I^*, w_I)$ increases with w_I; for any given w_I, $\Pi_I(Q_I^*, w_I)$ decreases with r_I; $\Omega_I(Q_I^*, w_I)$ and $\Gamma_I(Q_I^*, w_I)$ are unimodal functions in w_I. One additional observation is that for any wholesale price the manufacturer's profit decreases with the newly imposed interest rate (r_I), because the manufacturer's financial cost increases with the interest rate. In the following, we first compare in-house factoring financing to bank financing and then to early payment.

4.2 Comparison of in-house factoring to bank financing

To focus on in-house factoring financing, this subsection compares in-house factoring financing to banking financing by assuming that the retailer does not offer early payment. As in Lemma 4 and Lemma 5, for any $c \in (v, p]$, there exists a unique point $w_I^M(c)$ under in-house factoring financing, such that given $w_I^M(c)$ the manufacturer earns the same expected profit as that under bank financing. In addition, there exists a unique point $\dot{c}_I$ satisfying $\Gamma(\tilde{Q}) = \Omega_b^*$, such that, for any $c \in (v, \dot{c}_I)$, there exists a unique point $w_I^R(c) \in (w_I^*, p)$ where the retailer earns the same profit under in-house factoring financing as under bank financing. Therefore, there exists a Pareto zone $w_I \in [w_I^M, w_I^R]$, in which both the retailer and the manufacturer prefer in-house factoring financing to bank financing (i.e., $\Pi_I(Q_I^*, w_I) \geq \Pi_b^*$ and $\Omega_I(Q_I^*, w_I) \geq \Omega_b^*$). For its own benefit, the retailer mandates a wholesale price at $w_I^M(c)$.

Reasonably, the in-house factoring financing interest rate cannot be too high. We further define a threshold point $\hat{r}_I$, which satisfies $\Pi_I(Q_b^*, w_I, \hat{r}_I) = \Pi_b^*$ (where $w_I = p$). At this point, the wholesale price is at its highest level (i.e., $w_I = p$), so $\hat{r}_I$ is the upper bound of the interest rate. Beyond this threshold, the manufacturer always chooses bank financing over in-house factoring financing. Because the wholesale price in bank financing is no more than the retail price (i.e.,

$w_I \le p$), it is straightforward that $\hat{r}_I \ge r_b^*$.

Combining the above discussion on w_I^M and $\hat{r}_I$, as in Theorem 3, we can characterize a threshold policy for the retailer in selecting an equilibrium financing strategy between in-house factoring financing and bank financing.

Theorem 4 *[Financing equilibrium for in-house factoring financing] Suppose $0 < r_I \le \hat{r}_I$ for any $c \in (v, \dot{c}_I]$.*

1. *For any $c \in (v, \hat{c}_I]$, $w_I^M(r_I)$ increases with r_I; whereas $Q_I^*(w_I^M(r_I))$ decreases with r_I;*
2. *There exists a threshold point $\hat{c}_I \in (v, \dot{c}_I)$, where $\hat{c}_I$ satisfies $w_I^M(\hat{c}_I) = w_I^R(\hat{c}_I)$, such that both the retailer and the manufacturer are indifferent between bank financing and in-house factoring financing;*
3. *If $v < c < \hat{c}_I$, the unique sub-game perfect financing equilibrium is in-house factoring financing; otherwise (when $\hat{c}_I \le c \le p$), the equilibrium is bank financing.*

Theorem 4 (1) shows that when $c \in (v, \hat{c}_I]$, as the interest rate increases, the retailer must grant a higher wholesale price for the manufacturer for in-house factoring financing. However, because the wholesale price increment cannot recoup the losses caused by the higher interest rate, the manufacturer reduces the production quantity.

Theorem 4 further identifies such a financing equilibrium domain that in-house factoring financing outperforms bank financing for both the retailer and the manufacturer. This outcome resembles Theorem 3, except for r_I. If r_I is too big (i.e., $r_I > \hat{r}_I$), then it is intuitive that in-house factoring is not competitive anymore and gives way to bank financing.

4.3 Comparison of in-house factoring to early payment

It is intuitive that the retailer will offer either early payment or in-house factoring, whichever more profitable. As previously discussed, mathematically early payment is a special case of in-house factoring financing when the interest rate is zero and there is no setup cost. Given that the retailer can earn additional interest from in-house factoring, one might wonder whether the retailer would have incentives to give up its interest by setting $r_I = 0$. If the corner solution (i.e., $r_I = 0$) does occur, based on the tie-breaking rule stated in the Model, early payment is chosen; otherwise, positive-interest in-house factoring dominates (i.e., $r_I > 0$).

As it turns out, the relation between in-house factoring financing and early payment is more subtle than a monotonically increasing or decreasing r_I. We use the following outcome to first characterize the thresholds comparing either early payment or in-house factoring financing to bank financing.

Lemma 6 *Given any $r_I \in [0, \hat{r}_I]$, we have:*

1. $\dot{c}_I = \dot{c}_e$;
2. $\hat{c}_I \geq \hat{c}_e$.

To explain Lemma 6(1), we recall that $\dot{c}_e$ is the threshold point, at which the optimal wholesale price in early payment equals the production cost, so the manufacturer earns zero profit and the retailer earns the whole supply chain's profit. Similarly, $\dot{c}_I$ is the corresponding threshold point in in-house factoring financing. According to Theorem 3 and the preceding arguments leading to Theorem 4, the retailer earns the same profit in early payment or in-house factoring financing as in bank financing. Therefore, $\dot{c}_I = \dot{c}_e$, which implies the same upper bound for $\hat{c}_e$ and $\hat{c}_I$.

Nevertheless, we find that the retailer's profit is not always worse off when it has to raise the wholesale price to w_I^M. Indeed, for any given positive r_I, we have $\hat{c}_I > \hat{c}_e$ (Lemma 6(2)). This observation suggests that when the production cost is substantially high (i.e., $c \in (\hat{c}_e, \hat{c}_I)$), the retailer can no longer benefit from early payment compared with bank financing, but the retailer can still benefit from in-house factoring financing compared with bank financing. Therefore, Lemma 6(2) reveals that when $c \in (\hat{c}_e, \hat{c}_I)$, in-house factoring financing can outperform early payment for the retailer in financing equilibrium, which is discussed in more detail as follows.

Theorem 5 *[In-house factoring vs. early payment] For any $r_I \in (0, \hat{r}_I]$, there exist thresholds $\tilde{c}_3 \leq \tilde{c}_2 < \tilde{c}_1$, such that,*

1. *[Early payment dominates] If $c < \tilde{c}_3$, the retailer prefers early payment financing to in-house factoring financing (i.e., $\Omega_I(Q_I^*, w_I^M(r_I)) < \Omega_e(Q_e^*, w_e^M)$);*
2. *[Contingent equilibrium area] If $\tilde{c}_3 \leq c < \tilde{c}_2$, there exists an r_I^M (as illustrated in Figure 5) such that the retailer prefers in-house factoring financing to early payment if $r_I \leq r_I^M$ (i.e., $\Omega_I(Q_I^*, w_I^M(r_I)) \geq \Omega_e(Q_e^*, w_e^M)$); otherwise, the retailer prefers early payment (i.e., $\Omega_I(Q_I^*, w_I^M(r_I)) < \Omega_e(Q_e^*, w_e^M)$);*
3. *[In-house factoring financing dominates] If $\tilde{c}_2 \leq c < \tilde{c}_1$, the retailer prefers in-house factoring financing to early payment, and its profit firstly increases and then decreases with r_I;*
4. *[In-house factoring financing dominates] If $\tilde{c}_1 \leq c \leq \hat{c}_I$, the retailer prefers in-house factoring financing to early payment, and its profit monotonically increases with r_I.*

To fully explain Theorem 5, we identify several intertwining forces. From the retailer's perspective under in-house factoring financing, on the positive side, a positive interest rate generates extra financial income (*interest effect*). On the negative side, however, the interest rate forces the retailer's hand to provide a higher wholesale price to the manufacturer (*wholesale-price effect*);

meanwhile, the higher wholesale price is not able to recompense the additional financial cost for the manufacturer, such that the production quantity shrinks (*production-quantity effect*).

When the production cost is low (i.e., $c < \tilde{c}_3$, as illustrated in Figure 4(1) and the region of "E" of Figure 5 for Theorem 5(1)), both the retailer's and the manufacturer's profit margins are large. Consequently, both firms' profits are sensitive to production quantity change. Adding any extra cost to the production significantly curtails the manufacturer's production quantity. Provided that the production cost is small, the direct gain from the interest from in-house factoring financing (interest effect) is not enough to compensate for the revenue loss due to the reduced production quantity (production-quantity effect) and the higher wholesale price (wholesale-price effect); therefore, early payment dominates in-house factoring financing.

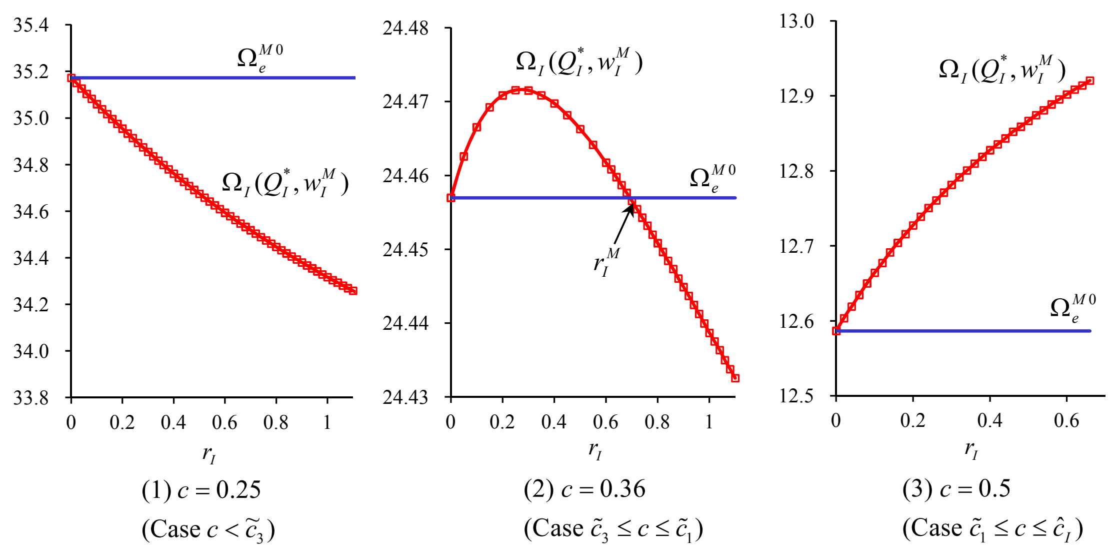


Figure 4: Impacts of c and r_I on the retailer's profit in early payment and in-house factoring financing.

As the production cost increases (i.e., $\tilde{c}_3 \leq c < \tilde{c}_2$, as illustrated in Figure 4(2) and the region of "I" of Figure 5 for Theorem 5(2)), the firms' profit margins dwindle, and both firms become less sensitive to production quantity change. As a result, the financial gain from the interest effect becomes more predominant, whereas the negative impact of the wholesale-price effect and production-quantity effect is restrained as long as the interest rate is not too high (i.e., $r_I \leq r_I^M$). In this situation, in-house factoring financing is still more attractive to the retailer, as depicted on the left side of Figure 5. However, if the interest rate is substantially high (i.e., $r_I > r_I^M$), the negative wholesale-price effect and production-quantity effect are relatively too enormous to overcome, such that early payment prevails (see the area of $\tilde{c}_3 \leq c < \tilde{c}_2$ in Figure 5).

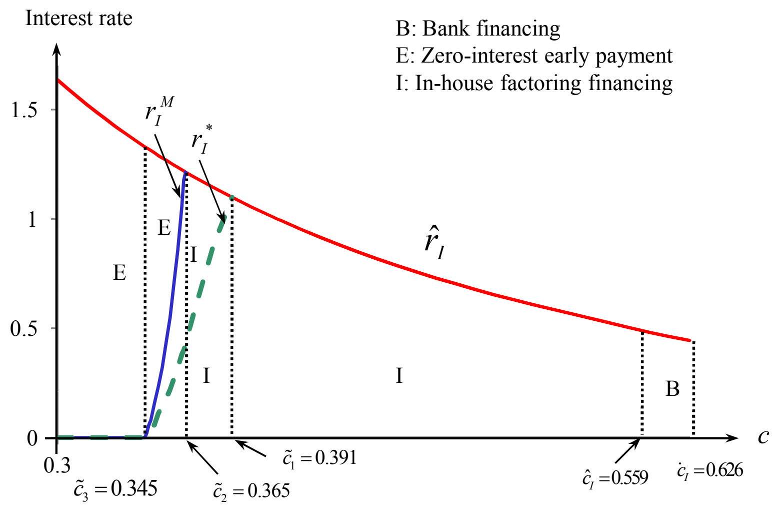


Figure 5: Equilibrium regions of zero-interest early payment (E) and in-house factoring financing (I).

As the production cost continues to grow (i.e., $\tilde{c}_2 \leq c < \tilde{c}_1$), the relative advantage of the positive interest effect against the negative wholesale-price effect and production-quantity effect expands, so the benefit of in-house factoring financing increases. In this scenario, similar to the previous scenario (i.e., $\tilde{c}_3 \leq c < \tilde{c}_2$), the retailer actually enjoys a higher interest rate. When the interest rate crosses a threshold, the negative wholesale-price effect and production-quantity effect escalate; however, the interest effect is so substantial that in-house factoring financing is always more desirable.

When the production cost is substantially high (i.e., $\tilde{c}_1 \leq c \leq \hat{c}_I$, as illustrated in Figure 4(3) and the region of "I" of Figure 5 for Theorem 5(4)), in-house factoring financing is dominant and the retailer has incentives to charge as high an interest rate as possible (i.e., r_I reaches the upper bound $\hat{r}_I$), because the financial gain from the higher interest overwhelms the operational revenue income.

The above phenomenon also reveals the interplay of the interest benefit of in-house factoring financing and the wholesale-price and production-quantity benefits in early payment. When the production cost is low, the wholesale-price and production-quantity benefits in early payment are more preeminent because of the larger profit margin and higher production quantity. However, as the production cost grows, the wholesale-price and production-quantity benefits in early payment diminish, while the interest benefit in in-house factoring financing is more commanding; thereupon, the retailer is more keen on reaping instant interest gains from in-house factoring financing.

Although we assume an external interest rate to focus on the factoring effect, we can actually characterize an optimal interest rate for in-house factoring financing as follows.

Theorem 6 *Consider in-house factoring financing.*

1. *When $\tilde{c}_3 < c < \tilde{c}_1$, the optimal interest rate $r_I^*(c)$ increases with c;*
2. *When $\tilde{c}_1 \leq c \leq \hat{c}_I$, the optimal interest rate reaches the upper bound (i.e., $r_I^*(c) = \hat{r}_I(c)$), and $\hat{r}_I(c)$) decreases with c.*

As shown in Theorem 6, when $\tilde{c}_3 \leq c < \tilde{c}_1$, there exists an optimal interest rate that is smaller than r_I^M and increases with the production cost, as illustrated in the dotted line of Figure 5. In this situation, as the production cost increases, the retailer has more incentives to collect instant interest and, thus, raise the optimal interest rate (i.e., $r_I^*(c)$). When $\tilde{c}_1 \leq c \leq \hat{c}_I$, the interest rate hits the upper bound ($\hat{r}_I$), because in-house factoring has to compete with bank financing to guarantee a reservation profit for the manufacturer. As the production cost grows, the retailer has to compromise by reducing the optimal interest rate to keep in-house factoring attractive.

As Figure 5 depicts, the retailer has incentives to set the interest rate to its minimal level (i.e., $r_I = 0$) under in-house factoring when the production cost is low (i.e., $c < \tilde{c}_3$). But, as previously explained, in-house factoring differs from early payment in multiple aspects. To the best of our knowledge, we have not observed zero-interest in-house factoring in practice, partially because of setup costs and license fees for processing factoring. Theorem 6 also suggests that the retailer should alter its interest rate according to the manufacturer's production cost, rather than fix it externally. Nevertheless, interest rate fixing is easier to implement and promotes the in-house factoring financing to a wide variety of manufacturers. It can also erase concerns of firm discrimination, such as that in the practice of JingDong Finance, which charges identical interest rates across different suppliers with in-house factoring (see, e.g., Chen et al., 2016).

4.4 The retailer's financing equilibrium selection

This section will compare all three financing schemes: bank financing, early payment, and in-house factoring financing using the information discussed above. Here we assume that all three financing schemes are viable. As the Stackelberg leader, the retailer can decide whether to offer early payment or in-house factoring and adjust the wholesale price accordingly in the pull system in its optimal financing scheme.

Theorem 7 *[Retailer's Equilibrium Financing Choice] Given that the retailer charges the optimal interest in in-house factoring financing, comparing bank financing, early payment, and in-house* *factoring financing results in the following outcome.*

1. *[Early payment dominates] If $v < c \le \tilde{c}_3$, the retailer prefers early payment financing;*
2. *[In-house factoring dominates] If $\tilde{c}_3 < c \le \hat{c}_I$, the retailer prefers in-house factoring financing;*
3. *[Bank financing dominates] If $\hat{c}_I < c \le p$, the retailer prefers bank financing.*

Theorem 7 summarizes the retailer's financing equilibrium choice among bank financing, early payment, and in-house factoring financing. Consistent with Theorem 5, Theorem 7 confirms that early payment continues to dominate in-house factoring financing when the production cost is sufficiently small (i.e., $v < c \le \tilde{c}_3$). Theorem 5 thus suggests that when $v < c \le \tilde{c}_3$ the retailer should not set up financing branch for in-house factoring but instead rely on early payment (see the left region of "E" in Figure 5).

As the production cost grows (i.e., $\tilde{c}_3 < c \le \hat{c}_I$), the wholesale price is higher, resulting in a lower production quantity, such that the retailer has incentives to charge a positive interest by switching from early payment to in-house factoring (see the middle region of "I" in Figure 5) by obtaining a license for and setting up a factoring financing branch. In our model, the cost to set up in-house factoring financing is assumed to be zero. In practice, as long as the cost to set-up in-house factoring is manageable, the retailer will opt for it.

Theoretically, however, it is not as straightforward for the retailer to shift away from early payment to in-house factoring financing, because the retailer can always command a lower wholesale price to compensate for the early payment. The retailer's shift between these two buyer financing schemes demonstrates the fundamentally different roles of wholesale pricing and interest rate in supply chain finance. In a pull supply chain, the manufacturer shoulders the inventory risk and is vulnerable to demand uncertainty. Provided that the manufacturer does not default, the manufacturer will earn $w_I \min[D, Q_I]$ for wholesaling in in-house factoring financing and pay $r_I c Q_I$ interest to the retailer. Thus, the wholesaling earnings are directly associated with demand uncertainty, while the interest is only indirectly affected by the uncertainty via production quantity. Correspondingly, the manufacturer has more incentives to produce more in early payment than in in-house factoring financing. If the manufacturer defaults, then the retailer will suffer from the losses from early payment or the in-house factoring loan. As the production cost increases, the default risk enhances. Therefore, in the range of $\tilde{c}_3 < c \le \hat{c}_I$, the retailer has incentives to shift to a less risky financing instrument and in-house factoring financing stands out.

For the same reason, when the production cost is considerably high, the manufacturer's default risk is also high, such that the retailer prefers the manufacturer to borrow from banks to finance its production (see the region of "B" in Figure 5).

5 Extended Discussions

This section firstly extends our model to a uniform wholesale price and then discusses the impact of demand variability.

5.1 A uniform wholesale price

Our baseline model has assumed that the retailer determines an optimal purchasing/wholesale price w_i^* for each respective financing scheme i. After a specific financing scheme is determined, the retailer can decide the wholesale price accordingly for each financing scheme. Our analysis shows that the wholesale price is lower in early payment than with bank financing, which is intuitive and has been observed in practice as an early payment discount (O'Connell, 2009). The retailer is likely to offer a higher purchasing (wholesale) price with in-house factoring, because the retailer earns extra interest from financing the manufacturer. In practice, however, it could be easier to implement a single identical wholesale price for different supply chain financing schemes. Thus, one might wonder whether the retailer can still benefit from early payment financing or in-house factoring even if the wholesale price is identical across all financing schemes. For theoretical comprehensiveness, this subsection provides a different perspective on how varying an identical wholesale price affects the financing equilibrium.

To demonstrate that our main qualitative results are robust even if the wholesale price is identical across different financing schemes, we assume that the identical wholesale price w_x can be any value in the feasible domain of $[c, p]$. In other words, w_x could be equal to one of the equilibrium wholesale prices in early payment, in-house factoring financing, and bank financing, or even something else. We use w_x^E to represent the wholesale price value that makes the manufacturer indifferent between bank financing and early payment, and w_x^I is the indifferent point between bank financing and in-house factoring.

When all financing schemes are available, it is intuitive that the manufacturer will choose the most beneficial financing scheme; however, as the financial service provider, the retailer can decide whether to offer early payment, in-house factoring, or none of them. We now explore the interaction between the manufacturer and the retailer and summarize the financing equilibrium in the following theorem.

Theorem 8 *Consider the case with an identical wholesale price w_x in bank financing, early payment and in-house factoring (i.e., $w_b = w_e = w_I = w_x$).*

1. *If $v < c \leq \check{c}$, then: when $c < w_x \leq w_x^E$, the financing equilibrium is early payment; when*

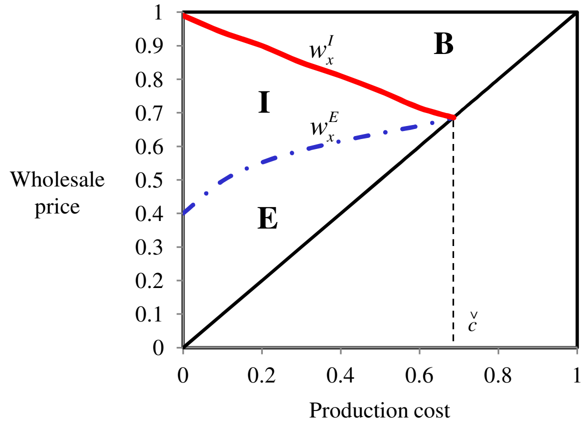
$w_x^E < w_x \leq w_x^I$, the financing equilibrium is in-house factoring; when $w_x^I < w_x \leq p$, the financing equilibrium is bank financing;

2. *If $\check{c} < c \leq p$, then the financing equilibrium is bank financing.*

Theorem 8 depicts a slightly different picture (illustrated in Figure 6) than that of Theorem 7 because we allow the identical wholesale price w_x to vary in the entire feasible domain. Hence, the fundamental disparity is that Theorem 8 assumes an identical wholesale price for all three financing schemes, whereas the retailer has the freedom to command different purchasing/wholesale prices in Theorem 7 for the baseline model. While the manufacturer always prefers a financing scheme with a lower interest rate when the wholesale price is fixed at the same level, as the Stackelberg leader, the retailer can still designate different wholesale price levels (identical for all financing schemes), and decide whether to offer either early payment or in-house factoring to dictate the final financing equilibrium outcome.

When the production cost is sufficiently low (i.e., $v < c \leq \check{c}$), the ultimate financing equilibrium relies on the wholesale price level. In this situation, the manufacturer's default risk is not too high. When the identical wholesale price is low (i.e., $c \leq w \leq w_x^E$), the final production quantity could suffer due to the low marginal profit for the manufacturer; therefore, the retailer has incentives to offer a zero-interest early payment to boost the manufacturer's production quantity, such that early payment emerges as the financing equilibrium (see the "E" region of Figure 6). When the identical wholesale price is in its medium level (i.e., $w_x^E < w \leq w_x^I$), the manufacturer's marginal profit is up and, thus, the manufacturer has more incentives to increase the production quantity. Correspondingly, the manufacturer's default risk increases, and the retailer then prefers in-house factoring by demanding interest to compensate for the higher default risk and its own lower profit margin. When the identical wholesale price is very high (i.e., $w > w_x^I$), the manufacturer's default risk is overwhelming for the retailer, such that the retailer prefers that the manufacturer utilizes bank financing.

In the above scenarios, because the manufacturer's default risk increases with the production cost, the region for early payment or in-house factoring financing as the financing equilibrium shrinks as the production cost grows (see Figure 6). For the same reason, when the production cost is considerably high (i.e., $\check{c} < c \leq p$), the retailer will avoid the default risk by offering no financing, forcing the manufacturer to borrow from bank financing. This qualitative observation is the same as that in Theorem 7 of the baseline model.



E: financing equilibrium: early payment scheme
I: financing equilibrium: in-house factoring scheme
B: financing equilibrium: bank financing

Figure 6: Financing equilibrium under a fixed wholesale price.

5.2 Impact of demand variability

To investigate the impact of demand variability, we hereby introduce a mathematical inequality, $J(\tilde{Q}) \geq 1 + 2\bar{F}(\tilde{Q})$, which represents the situation where demand variation is relatively large. If $J(\tilde{Q}) < 1 + 2\bar{F}(\tilde{Q})$, the demand variation is relatively low. Due to limited space, we focus on only early payment versus bank financing, and the analysis on in-house factoring financing can be done similarly.

Comparing the optimal production quantity between early payment and bank financing, we obtain the following lemma.

Lemma 7 *1. If demand variability is relatively high (i.e., $J(\tilde{Q}) \geq 1+2\bar{F}(\tilde{Q})$), we have $Q_e^* \geq Q_b^*$ for any $c \in (v, p]$;*

2. If demand variability is relatively low (i.e., $J(\tilde{Q}) < 1 + 2\bar{F}(\tilde{Q})$), when $c < \bar{c}_b$, $Q_e^* \leq Q_b^*$; and when $c \in (\bar{c}_b, p]$, $Q_e^* > Q_b^*$.

Lemma 7 exhibits that the production quantity in early payment is always higher than that in bank financing when the demand variability is sufficiently high (see Figure 7, left subfigure). This occurs because, as explained in Section 3.2, early payment is a better risk-sharing mechanism than bank financing for the supply chain. Nevertheless, the relative advantage of a higher production quantity is subjugated when the production cost is sufficiently small, so the retailer has more incentives to squeeze a higher profit margin from the manufacturer, leading to a lower production quantity in early payment (see Figure 7, right subfigure when $c < 0.37$).

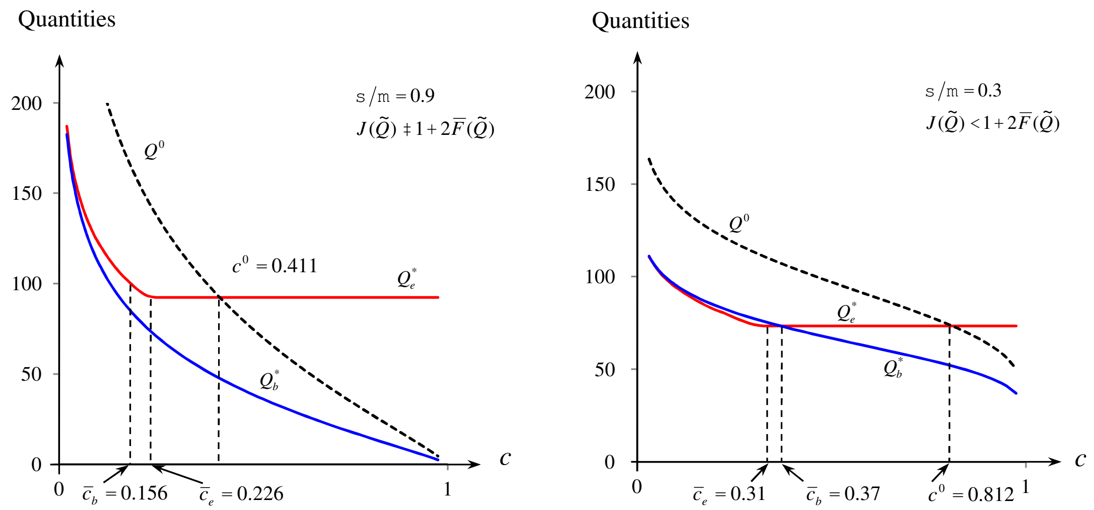


Figure 7: Comparison of production quantities between early payment and bank financing.

The demand variability naturally affects the total supply chain profit. Compared with Theorem 2, the following outcome demonstrates that the total supply chain profit advantage in early payment is more sensitive to demand variability, compared with that of the retailer.

Theorem 9 *There exists a threshold point* $\ddot{c}_e \in (\max\{\bar{c}_e, \bar{c}_b\}, \dot{c}_e)$ *where* $\Gamma_e^*(\ddot{c}_e) = \Gamma_b^*(\ddot{c}_e)$*, such that:*

1. *When* $J(\tilde{Q}) \geq 1 + 2\bar{F}(\tilde{Q})$*: if* $c \in (v, \ddot{c}_e]$*, then* $\Gamma_e^* \geq \Gamma_b^*$*; if* $c \in (\ddot{c}_e, p]$*,* $\Gamma_e^* < \Gamma_b^*$*;*
2. *When* $J(\tilde{Q}) < 1 + 2\bar{F}(\tilde{Q})$*: if* $c \in (v, \bar{c}_b)$*, then* $\Gamma_e^* < \Gamma_b^*$*; if* $c \in [\bar{c}_b, \ddot{c}_e]$*, then* $\Gamma_e^* \geq \Gamma_b^*$*; if* $c \in (\ddot{c}_e, p]$*,* $\Gamma_e^* < \Gamma_b^*$*.*

The twist in Theorem 9 occurs when the production cost is sufficiently low. Note that the manufacturer's financing scheme preference is not always aligned with the retailer's preference. When the demand variation is high (i.e., $J(\tilde{Q}) \geq 1 + 2\bar{F}(\tilde{Q})$), the retailer charges a higher risk premium for providing early payment; as a result, the retailer's gain in early payment surpasses the manufacturer's loss, such that the whole supply chain performs better with early payment when the production cost is low. When the demand variation is low (i.e., $J(\tilde{Q}) < 1 + 2\bar{F}(\tilde{Q})$), the retailer's gain in early payment can no longer make up the manufacturer's loss, such that the whole supply chain can benefit from bank financing especially when the production cost is sufficiently low.

To inspect how demand variability influences the efficiency improvement of early payment compared with bank financing, we further use $\frac{\Omega_e^* - \Omega_b^*}{\Omega_b^*}(\%)$ to represent the retailer's efficiency improvement when switching from bank financing to early payment financing. As Figure 8 depicts, the threshold value of $\hat{c}_e$ decreases as the coefficient variance of demand changes from $\sigma/\mu = 0.3$ to 0.9. This observation intuitively suggests that the region of early payment as the financing equilib-

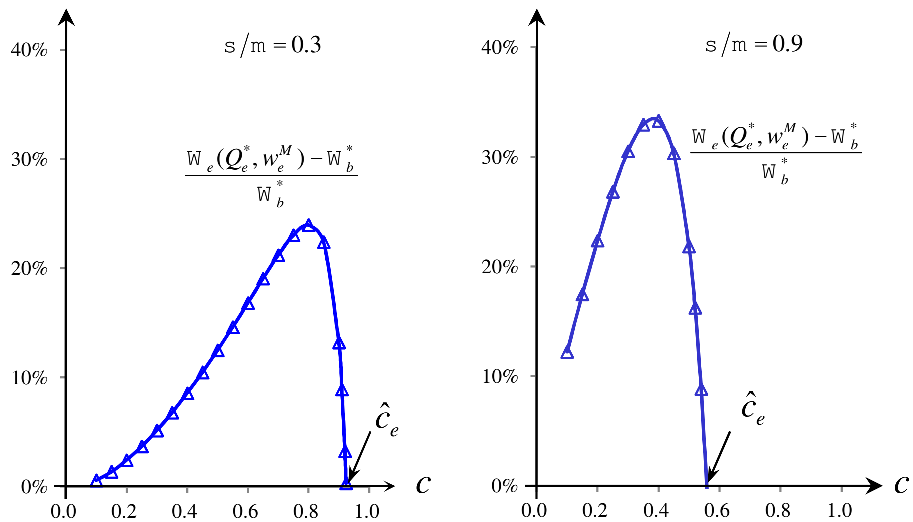


Figure 8: The retailer's efficiency improvement as demand variability increases.

rium shrinks as the demand variability grows. More interestingly, Figure 8 also exhibits that the retailer's improvement in efficiency in early payment first concavely increases and then decreases with the production cost. When the production cost is low, in early payment, the retailer can benefit more from the supply chain's risk-sharing mechanism than the damage of over-production from the manufacturer.

6 Conclusions

This paper investigates the efficacy of no-interest early payment financing and positive-interest in-house factoring financing in a pull supply chain with a capital-constrained manufacturer selling through a retailer. We first characterize the optimal solutions in both bank financing and early payment financing. The comparison of these two financing schemes shows that as long as the manufacturer's production cost is not too high, the retailer will strictly prefer early payment financing to bank financing. The early payment financing demonstrates a coordination effect on the supply chain because of the embedded risk-sharing mechanism between the retailer and the manufacturer. However, the retailer has incentives to command a lower wholesale price in early payment, so both the manufacturer and the whole supply chain may suffer.

However, to compete with bank financing, the retailer has to raise the wholesale price to attract the manufacturer to employ early payment. Although the retailer's profit drops due to the competition from bank financing, there exists a wholesale price Pareto zone, such that both firms are willing to implement early payment financing as long as the manufacturer's production cost is sufficiently low.

We further study in-house factoring financing, in which the retailer charges positive interest and essentially turns advance payment into a loan. Comparing in-house factoring financing with bank financing, we find that the financing equilibrium domain of in-house factoring financing can be larger than that of early payment. This finding reveals that in-house factoring financing can outperform early payment for the retailer and, thus, the whole supply chain. Particularly when the production cost is low, early payment without interest is more attractive, which indicates the advantage of the upfront commitment of no interest in early payment as compared to positive-interest in-house factoring. However, as the production cost grows, the preference for in-house factoring financing also increases and outpaces early payment financing. But, if the production cost is substantially high, it is profitable for the retailer to charge the interest rate at its upper bound, which decreases with the production cost.

Our extended analysis indicates that the main qualitative finding continues to hold when the retailer commands an identical wholesale price across all three financing schemes. We also demonstrate that, as demand variability increases, the production quantity is more likely to be higher in early payment than in bank financing, but the financing equilibrium domain of early payment shrinks.

This paper conveys three major managerial insights. First, this paper portrays a threshold policy on when to implement early payment financing compared with bank financing. It thus theoretically validates the reasoning for using early payment in practice, even if there is no interest charged on early payment. Second, we further manifest that it could be beneficial for the downstream firm to impose positive interest on advance payment via in-house factoring financing even if there is a setup cost. Our theory provides a guideline on whether to demand interest on advance payments and, if yes, under what conditions. Third, the implementation of early payment or in-house factor financing also depends on other factors, such as production cost and demand variability. Therefore, the managers in charge of early payment or in-house factoring financing should evaluate all these factors before implementing concrete financing terms.

This paper has its limitations and can be extended in several directions. First, due to computational complexity, we cannot analytically characterize a contract menu in the presence of asymmetric information, although we can numerically show that, with information asymmetry, the financing equilibrium domain becomes smaller. This research direction remains our top priority. Second, for tractability, the initial capital is assumed to be zero, which can be relaxed in future studies. Lastly, for tractability, we have assumed that the manufacturer will borrow either bank credit or buyer financing but not both at the same time. In practice, it is possible for the manufacturer to borrow both, especially if the retailer's initial capital is not sufficient to cover all the manufacturer's financial need.

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Appendix: Online Supplements

Proofs for "Buyer Financing in Pull Supply Chains: Zero-Interest Early Payment or In-House Factoring?"

Proof of Lemma 1. (1) Submit $w_b = (c-v)/\bar{F}(Q_b)+v$ into the retailer's payoff function $\Omega_b(w_b) = (p-w_b)S(Q_b)$, we can rewrite the retailer's payoffs function as $\Omega_b(Q_b) = [(p-v) - \frac{c-v}{\bar{F}(Q_b)}]S(Q_b)$. We then get $\frac{d\Omega_b(Q_b)}{dQ_b} = (p-v)\bar{F}(Q_b) - (c-v)J(Q_b)$. Since $\left[S(Q_b)/\bar{F}(Q_b)\right]' = J(Q_b) > 0$, we know $S(Q_b)/\bar{F}(Q_b)$ increases in Q_b. With the IFR property, $h(Q_b)$ increases in Q_b. Thus, we know $J(Q_b) = 1 + h(Q_b)S(Q_b)/\bar{F}(Q_b)$ increases with Q_b. Combining with that $\bar{F}(Q_b)$ decreases with Q_b, we know that $d\Omega_b/dQ_b$ decreases with Q_b. Thus, we know that $\Omega_b(Q_b)$ is a unimodal function in $Q_b \geq 0$. When $Q_b = 0$, we have $d\Omega_b/dQ_b = p-c > 0$, and $Q_b = +\infty$, $d\Omega_b/dQ_b < 0$. Thus, the optimal Q_b^* is solved by $(p-v)\bar{F}(Q_b^*) = (c-v)J(Q_b^*)$.

(2) Obviously, we have $w_b^* = (c-v)/\bar{F}(Q_b^*) + v$.

(3) We denote, $T(Q_b^*) = \frac{(1+r_b^*)c-v}{c-v}V(Q_b^*)$. From Eq. (1) and $w_b^* = (c-v)/\bar{F}(Q_b^*)+v$, we know, the bank's optimal interest rate satisfies $\frac{c-v}{\bar{F}(Q_b^*)}\int_0^{T(Q_b^*)} DdF(D) + \ (c(1+r_b^*)-v)Q_b^*\bar{F}(T(Q_b^*)) = (c-v)Q_b^*$. After simplifying the above equation, we get $S\left(T(Q_b^*)\right) = V(Q_b^*)$. Furthermore, we know $S(Q) = \mathbb{E}\min[D,Q] \leq Q$, and so $T(Q_b^*) \geq S\left(T(Q_b^*)\right) = V(Q_b^*)$. Thus, $\frac{(1+r_b^*)c-v}{c-v} \geq 1$, and then we have $r_b^* \geq 0$. Q.E.D.

Proof of Corollary 1. With Lemma 1, we get the retailer's and the manufacturer's optimal expected payoffs, $\Omega_b^* = \left[p - v - \frac{c-v}{\bar{F}(Q_b^*)}\right]S(Q_b^*)$, $\Pi_b^* = \frac{c-v}{\bar{F}(Q_b^*)}[S(Q_b^*) - Q_b^*\bar{F}(Q_b^*)]$. The bank obtains an expected net profit $r_f c Q_b^* = 0$, and the total profit in the supply chain with bank loans is $\Gamma_b^* = (p-v)S(Q_b^*) - (c-v)Q_b^*$. From Lemma 1, $\partial\Omega_b^*/\partial Q_b^* = 0$, we have, $\frac{d\Omega_b^*}{dc} = \frac{\partial\Omega_b^*}{\partial Q_b^*}\frac{dQ_b^*}{dc} + \frac{\partial\Omega_b^*}{\partial c} = \frac{\partial\Omega_b^*}{\partial c} = -\frac{S(Q_b^*)}{\bar{F}(Q_b^*)} < 0$. Obviously, we have $dQ_b^*/dc < 0$ and $Q_b^* < Q^0$. Then, when $c > v$, we have $(p-v)\bar{F}(Q_b^*) - (c-v) > (p-v)\bar{F}(Q^0) - (c-v) = 0$. Thus, $d\Gamma_b^*(c)/dc = ((p-v)\bar{F}(Q_b^*) - (c-v))dQ_b^*/dc - Q_b^* < 0$. Q.E.D.

Proof of Lemma 2. (1) Taking derivative of Q_e in Eq. (3), we can get, $\frac{d\Pi_e}{dQ_e} = (w_e - v)\bar{F}(Q_e) - (c-v)\bar{F}\left(L_e(Q_e)\right)$. When $w_e > c$, we have $L_e(Q_e) < Q_e$, with the IFR property, $\frac{d^2\Pi_e}{dQ_e^2} = (w_e - v)\bar{F}(Q_e)[\frac{c-v}{w_e-v}h(L_e(Q_e)) - h(Q_e)] < 0$. Then, when $w_e > c$, the optimal production quantity Q_e^* can be solved by $(w_e - v)\bar{F}(Q_e^*) = (c-v)\bar{F}(L_e(Q_e^*))$.

Now, we consider the properties of function $V(Q) = Q\bar{F}(Q)$. We have $V'(Q) = \bar{F}(Q)(1-H(Q))$. With our assumption of IFR property on demand, we know $H(Q)$ is increasing in Q. Then $1-H(Q)$ is decreasing in Q. Thus $V'(Q)$ is decreasing in Q. Thus, the function $V(Q)$ is a unimodal function and the point $\tilde{Q}$ is the maximum point. Furthermore, if $Q > \tilde{Q}$, $V(Q)$ decreases with Q; otherwise,

$V(Q)$ increases with Q.

With the relation $(w_e - v)\bar{F}(Q_e^*) = (c-v)\bar{F}(L_e(Q_e^*))$, and $L_e(Q_e^*) = \frac{c-v}{w_e-v}Q_e^*$, we get $Q_e^*\bar{F}(Q_e^*) = L_e(Q_e^*)\bar{F}(L_e(Q_e^*))$. So, the optimal production quantity Q_e^* satisfies $V(Q_e^*) = V(L_e(Q_e^*))$. Since the function $V(Q)$ is a unimodal function, we must have $L_e(Q_e^*) \leq \tilde{Q} \leq Q_e^*$. With the IFR property, we know $H(L_e(Q_e^*)) \leq H(\tilde{Q}) \leq H(Q_e^*)$ and $H(\tilde{Q}) = 1$. Therefore, when $w_e = c$, $Q_e^* = \tilde{Q}$.

(2) From IFR property, $h(Q_e^*) > h(L_e(Q_e^*))$. Thus, from $(w_e - v)\bar{F}(Q_e^*) = (c-v)\bar{F}(L_e(Q_e^*))$, we have that,

$$\frac{dQ_e^*}{dw_e} = \frac{1 - H(L_e(Q_e^*))}{(w_e - v)h(Q_e^*) - (c-v)h(L_e(Q_e^*))} > 0 \tag{A-1}$$

and then $Q_e^* \geq \tilde{Q}$. Q.E.D.

Proof of Lemma 3. (1) For any $w_e \in (c,p]$, we have $\frac{d\Pi_e(Q_e^*,w_e)}{dw_e} = \frac{\partial \Pi_e(Q_e^*,w_e)}{\partial Q_e^*}\frac{dQ_e^*}{dw_e} + \frac{\partial \Pi_e(Q_e^*,w_e)}{\partial w_e} = Y(Q_e^*) - Y(L_e(Q_e^*)) + Q_e^*\bar{F}(Q_e^*)$. Function $Y(Q) = \int_0^Q DdF(D)$ is increasing in Q. So, with the continuity of $\Pi_e(Q_e^*, w_e)$ in $w_e \in (c,p]$, we know the manufacturer's profit $\Pi_e(Q_e^*, w_e)$ increases with $w_e \in (c,p]$.

(2) For any $w_e \in (c,p]$ in Eq. (5), we get $\frac{\partial \Omega_e(Q_e^*,w_e)}{\partial Q_e^*} = (p-v)\bar{F}(Q_e^*) - (c-v)$, $\frac{\partial \Omega_e(Q_e^*,w_e)}{\partial w_e} = -S(Q_e^*) + Y(L_e(Q_e^*))$. Consequently, we get

$$\begin{aligned}\frac{d\Omega_e(Q_e^*,w_e)}{dw_e} &= \frac{\partial \Omega_e(Q_e^*,w_e)}{\partial Q_e^*}\frac{dQ_e^*}{dw_e} + \frac{\partial \Omega_e(Q_e^*,w_e)}{\partial w_e} \\ &= ((p-v)\bar{F}(Q_e^*) - (c-v)) \cdot \frac{Q_e^*}{w_e - v} \cdot (1 - M(Q_e^*)) - S(Q_e^*) + Y(L_e(Q_e^*)). \end{aligned} \tag{A-2}$$

From Lemma 10, we know $1 - M(Q_e^*)$ decreases with w_e. We know, $\frac{d(Q_e^*/(w_e-v))}{dw_e} = -\frac{Q_e^* M(Q_e^*)}{(w_e-v)^2} < 0$. Obviously, $(p-v)\bar{F}(Q_e^*) - (c-v)$ decreases with w_e. Also, with the results $dQ_e^*/dw_e > 0$ and $dL_e(Q_e^*)/dw_e < 0$ (from (A-10)), the functions $-S(Q_e^*)$ and $Y(L_e(Q_e^*))$ decreases with w_e. Thus, we know $d\Omega_e/dw_e$ decreases in w_e, which means that $\Omega_e(Q_e^*, w_e)$ is a unimodal function in $w_e \in (c,p]$.

(3) Also, from the above results, we have $d\Gamma_e(Q_e^*, w_e)/dw_e = ((p-v)\bar{F}(Q_e^*) - (c-v))\frac{dQ_e^*}{dw_e} = ((p-v)\bar{F}(Q_e^*) - (c-v)) \cdot \frac{Q_e^*}{w_e-v} \cdot (1 - M(Q_e^*))$ is a decreasing function in w_e, which means that $\Gamma_e(Q_e^*, w_e)$ is a unimodal function in $w_e \in (c,p]$. Q.E.D.

Proof of Theorem 1. (1) First, for any given w_e, we denote $K(Q_e^*) = -V(Q_e^*)U(Q_e^*) - (Y(Q_e^*) - Y(L(Q_e^*)))$, where

$$U(Q_e^*) = 1 - (1 - M(Q_e^*))\frac{(p-v)\bar{F}(Q_e^*) - (c-v)}{(w_e - v)\bar{F}(Q_e^*)}. \tag{A-3}$$

Then, from Eq. (A-2), we get $d\Omega_e(Q_e^*, w_e)/dw_e = K(Q_e^*)$.

From Lemma 10, we have $0 \leq 1 - M(Q_e^*) \leq 1/2$. Obviously, we have $\frac{(p-v)\bar{F}(Q_e^*)-(c-v)}{(p-v)\bar{F}(Q_e^*)} < 1$. Then, when $w_e = p$, $(1 - M(Q_e^*))\frac{(p-v)\bar{F}(Q_e^*)-(c-v)}{(p-v)\bar{F}(Q_e^*)} \leq 1/2$, and then $U(Q_e^*) > 1/2$. Furthermore, for any w_e, we have $V(Q_e^*) > 0$, $Y(Q_e^*) - Y(L_e(Q_e^*)) \geq 0$. Thus, when $w_e = p$, $K(Q_e^*) = -V(Q_e^*)U(Q_e^*) - (Y(Q_e^*) - Y(L_e(Q_e^*))) < 0$.

From Lemma 10, if $w_e = c$, then $M(Q_e^*) = 1/2$, and $Q_e^* = \tilde{Q}$, $L_e(Q_e^*) = \tilde{Q}$, $Y(Q_e^*) - Y(L_e(Q_e^*)) =$

0. Thus, when $w_e = c$, we have $K(Q_e^*|w_e = c) = -V(\tilde{Q})\left(1 - \frac{(p-v)\bar{F}(\tilde{Q})-(c-v)}{2(c-v)\bar{F}(\tilde{Q})}\right)$. From the definition of $\bar{c}_e$, we can verify that: if $c \leq \bar{c}_e$, then $K(Q_e^*|w_e = c) \geq 0$; if $c > \bar{c}_e$, then $K(Q_e^*|w_e = c) < 0$.

From the proof of Lemma 3, we know that $K(Q_e^*)$ decreases with w_e. Thus, there are two sceneries for the optimal wholesale price w_e^*: 1) Scenario $\bar{c}_e < c \leq p$: From above, there is $K(Q_e^*|w_e = c) \leq 0$. Thus, for all $w_e \in (c, p]$, we have $d\Omega_e(Q_e^*, w_e)/dw_e < 0$. The optimal wholesale price is $w_e^* = c$; 2) Scenario $v < c \leq \bar{c}_e$. From above, there is $K(Q_e^*|w_e = c) > 0$ and $K(Q_e^*|w_e = p) < 0$. So, there must exist one $w_e^* \in (c, p]$ satisfying $K(Q_e^*|w_e = w_e^*) = 0$, i.e., $d\Omega_e(Q_e^*, w_e^*)/dw_e^* = 0$. Then, w_e^* is the optimal wholesale price.

(2) Now, we prove that, under the equilibrium Q_e^*, w_e^*, we have $dQ_e^*(c)/dc \leq 0$ and $d\Omega_e^*(c)/dc < 0$ in both domains $c \in (v, \bar{c}_e)$ and $c \in [\bar{c}_e, p]$, respectively.

We first prove that the above results in the domain of $c \in [\bar{c}_e, p]$. When $c \in [\bar{c}_e, p]$, from part (1), we obtain $w_e^* = c$ and $Q_e^*(c) = \tilde{Q}$, and $\Omega_e^*(c) = (p - v)S(\tilde{Q}) - (c - v)\tilde{Q}$. Then, we have $dQ_e^*(c)/dc = 0$ and $d\Omega_e^*(c)/dc = -\tilde{Q} < 0$ in $c \in [\bar{c}_e, p]$.

Now we use contradiction approach to prove $dQ_e^*(c)/dc \leq 0$ and $d\Omega_e^*(c)/dc < 0$ in the domain of $c \in (v, \bar{c}_e)$. To prove that, We hereby suppose $dQ_e^*(c)/dc > 0$ for some $c \in (v, \bar{c}_e)$.

Before we conduct the contradiction approach to prove the results, we analyze the properties of $dL_e(Q_e^*(c))/dc$, $dM(Q_e^*(c))/dc$ and $\partial Q_e^*(c, w_e^*)/\partial c$ in advance.

Firstly, we analyze the property of $dL_e(Q_e^*(c))/dc$. From Lemma 2, the Q_e^* and w_e^* satisfy $(w_e^* - v)\bar{F}(Q_e^*) = (c-v)\bar{F}(L_e(Q_e^*))$, where $L_e(Q_e^*) = \frac{(c-v)Q_e^*}{w_e^*-v}$. Then, we multiple both sides with $Q_e^*/(w_e^* - v)$, we have $Q_e^*\bar{F}(Q_e^*) = L_e(Q_e^*)\bar{F}(L_e(Q_e^*))$. By doing total derivative on both sides of this equation, we have, $\bar{F}(Q_e^*)\frac{dQ_e^*(c)}{dc} - Q_e^* f(Q_e^*)\frac{dQ_e^*(c)}{dc} = \bar{F}(L_e(Q_e^*))\frac{dL_e(Q_e^*)}{dc} - L_e(Q_e^*)f(L_e(Q_e^*))\frac{dL_e(Q_e^*)}{dc}$. Reorganizing the equation giving us $\frac{dL_e(Q_e^*)}{dc} = \frac{\bar{F}(Q_e^*)-Q_e^*f(Q_e^*)}{\bar{F}(L_e(Q_e^*))-L_e(Q_e^*)f(L_e(Q_e^*))} \cdot \frac{dQ_e^*(c)}{dc} = -\frac{L_e(Q_e^*)}{Q_e^*} \cdot \frac{H(Q_e^*)-1}{1-H(L_e(Q_e^*))} \cdot \frac{dQ_e^*(c)}{dc}$. Recall that, from the proof of proof of Lemma 2, we have $H(L_e(Q_e^*)) < 1 < H(Q_e^*)$. Therefore, with the assumption $dQ_e^*(c)/dc > 0$, we obtain $dL_e(Q_e^*(c))/dc < 0$.

Secondly, we analyze the property of $dM(Q_e^*(c))/dc$. From the definition of $M(Q_e^*)$ in Lemma 10 and $\frac{dL_e(Q_e^*)}{dc}$ in the preceding paragraph, we have,

$$
\begin{aligned}
\frac{dM(Q_e^*)}{dc} &= \frac{H'(Q_e^*)\frac{dQ_e^*(c)}{dc}(1 - H(L_e(Q_e^*))) - H'(L_e(Q_e^*))(H(Q_e^*) - 1)\frac{L_e(Q_e^*)}{Q_e^*} \cdot \frac{H(Q_e^*)-1}{1-H(L_e(Q_e^*))} \cdot \frac{dQ_e^*(c)}{dc}}{(H(Q_e^*) - H(L_e(Q_e^*)))^2} \\
&= \frac{1 - H(L_e(Q_e^*))}{(H(Q_e^*) - H(L_e(Q_e^*)))^2} \cdot \frac{dQ_e^*(c)}{dc}\left[H'(Q_e^*) - H'(L_e(Q_e^*))\frac{L_e(Q_e^*)}{Q_e^*} \cdot \frac{(H(Q_e^*) - 1)^2}{(1 - H(L_e(Q_e^*)))^2}\right] \\
&= \Theta(w_e^*)\frac{1 - H(L_e(Q_e^*))}{(H(Q_e^*) - H(L_e(Q_e^*)))^2} \cdot \frac{dQ_e^*(c)}{dc},
\end{aligned}
$$

where $\Theta(w_e^*) = H'(Q_e^*) - H'(L_e(Q_e^*))\frac{L_e(Q_e^*)}{Q_e^*} \cdot \frac{(H(Q_e^*)-1)^2}{(1-H(L_e(Q_e^*)))^2}$ is defined in the proof of Lemma 10. From the proof of Lemma 10, for any $w_e \in (c, p]$, $\Theta(w_e) > 0$. So, we must have $\Theta(w_e^*) > 0$. Thus, with the assumption $dQ_e^*(c)/dc > 0$, we have $\frac{dM(Q_e^*)}{dc} > 0$.

Thirdly, we analyze the property of $\frac{\partial Q_e^*(c,w_e^*)}{\partial c}$. Because, for any w_e, $L_e(Q_e^*(c)) = \frac{c-v}{w_e-v}Q_e^*(c,w_e)$, then, for any fixed w_e, taking partial derivatives on both sides, we obtain $\frac{\partial L_e(Q_e^*(c))}{\partial c} = \frac{Q_e^*(c)}{w_e-v} + \frac{c-v}{w_e-v}\frac{\partial Q_e^*(c)}{\partial c}$. So, for any given w_e, from $(w_e-v)\bar{F}(Q_e^*) = (c-v)\bar{F}(L_e(Q_e^*))$, taking partial derivatives on both sides, we have $-(w_e-v)f(Q_e^*(c,w_e))\frac{\partial Q_e^*(c,w_e)}{\partial c} = \bar{F}(L_e(Q_e^*(c,w_e)))-(c-v)f(L_e(Q_e^*(c,w_e)))$ $\left(\frac{Q_e^*(c,w_e)}{w_e-v} + \frac{c-v}{w_e-v}\frac{\partial Q_e^*(c,w_e)}{\partial c}\right)$. After rearranging this equation, we get $\frac{\partial Q_e^*(c,w_e)}{\partial c} = -\frac{Q_e^*(1-H(L_e(Q_e^*)))}{(c-v)[H(Q_e^*)-H(L_e(Q_e^*))]}$. So, from the proof of Lemma 2, $H(L_e(Q_e^*)) < 1 < H(Q_e^*)$, we have $\frac{\partial Q_e^*(c,w_e)}{\partial c} < 0$. When $w_e = w_e^*$, we also have $\frac{\partial Q_e^*(c,w_e^*)}{\partial c} < 0$.

We now use the contradiction approach. On the one hand, from (A-1) in the proof of Lemma 2, for any fixed c, we have $\frac{\partial Q_e^*(c,w_e)}{\partial w_e} > 0$. We further have $\frac{dQ_e^*(c)}{dc} = \frac{dQ_e^*(c,w_e^*)}{dc} = \frac{\partial Q_e^*(c,w_e^*)}{\partial w_e} \cdot \frac{dw_e^*}{dc} + \frac{\partial Q_e^*(c,w_e^*)}{\partial c}$. Given that $\frac{dQ_e^*(c)}{dc} > 0$, $\frac{\partial Q_e^*(c,w_e^*)}{\partial c} < 0$, $\frac{\partial Q_e^*(c,w_e^*)}{\partial w_e} > 0$, we must have $\frac{dw_e^*}{dc} = \left(\frac{dQ_e^*(c)}{dc} - \frac{\partial Q_e^*(c,w_e^*)}{\partial c}\right)/\frac{\partial Q_e^*(c,w_e^*)}{\partial w_e} > 0$ in the domain of $c \in (v, \bar{c}_e)$.

On the other hand, we consider the property of $K(Q_e^*(c), w_e^*(c))$ with regard to c given the equilibrium $Q_e^*(c)$ and $w_e^*(c)$. From the proof of Lemma 2, we know $V(Q) = Q\bar{F}(Q)$ is decreasing in Q when $Q \geq \tilde{Q}$. We already know $Q_e^*(c) \geq \tilde{Q}$. Thus, with the assumption $dQ_e^*(c)/dc > 0$, we must have $V(Q_e^*(c)) = Q_e^*(c)\bar{F}(Q_e^*(c))$ decreasing in c. Also, from $dQ_e^*(c)/dc > 0$, $(c-v)Q_e^*(c)$ increases in c. Hence, $(p-v)Q_e^*\bar{F}(Q_e^*(c)) - (c-v)Q_e^*(c)$ is decreasing in c. From the property explained previously, we know $M(Q_e^*(c))$ is increasing in c. So, the function $1 - M(Q_e^*(c))$ is decreasing in c. Obviously, the expected sale, $S(Q)$, increases with Q. So, if $dQ_e^*(c)/dc > 0$, the function $S(Q_e^*(c))$ is increasing in c. The function $Y(Q) = \int_0^Q DdF(D)$ is increasing in Q. Thus, from the previous property $dL_e(Q_e^*)/dc < 0$, we can infer that the function $Y(L_e(Q_e^*(c)))$ is decreasing in c. From Part (1) in the above, we know, when $v < c \leq \bar{c}_e$, the optimal $w_e^*(c)$ satisfies the first-order condition $K(Q_e^*(c), w_e^*(c)) = 0$, that is, $K(Q_e^*(c), w_e^*(c)) = \frac{(p-v)Q_e^*\bar{F}(Q_e^*(c))-(c-v)Q_e^*(c)}{w_e^*(c)-v} \cdot (1-M(Q_e^*(c))) - S(Q_e^*(c)) + Y(L_e(Q_e^*(c))) = 0$. Solving $K(Q_e^*(c), w_e^*(c)) = 0$, we have $w_e^*(c) = v + \frac{(p-v)Q_e^*\bar{F}(Q_e^*(c))-(c-v)Q_e^*(c)}{S(Q_e^*(c))-Y(L_e(Q_e^*(c)))} \cdot (1-M(Q_e^*(c)))$. Thus, from the above results, we get that $\frac{(p-v)Q_e^*\bar{F}(Q_e^*(c))-(c-v)Q_e^*(c)}{S(Q_e^*(c))-Y(L_e(Q_e^*(c)))} \cdot (1-M(Q_e^*(c)))$ is decreasing in c. Consequently, we know w_e^* is decreasing in c (i.e., $\frac{dw_e^*}{dc} \leq 0$), which is contradictory to the conclusion of $\frac{dw_e^*(c)}{dc} > 0$ concluded in the preceding paragraph for the same assumption (i.e., $dQ_e^*(c)/dc > 0$). Therefore, the assumption $dQ_e^*(c)/dc > 0$ is problematic for some $c \in (v, \bar{c}_e)$. As a result, we must have $dQ_e^*(c)/dc \leq 0$ for all $c \in (v, \bar{c}_e)$.

At last, we prove $\frac{d\Omega_e^*(c)}{dc} < 0$ when $c \in (v, \bar{c}_e)$. From part (1), we have $K(Q_e^*) = 0$. Then, from Eq. (A-2), there must have $((p-v)\bar{F}(Q_e^*) - (c-v)) \cdot \frac{Q_e^*}{w_e^*-v} \cdot (1-M(Q_e^*)) = S(Q_e^*) - Y(L_e(Q_e^*)) = Q_e^*\bar{F}(Q_e^*) + Y(Q_e^*) - Y(L_e(Q_e^*)) > 0$. Furthermore, from Lemma 10, we have $0 \leq 1 - M(Q_e^*) \leq 1/2$. So, we have $(p-v)\bar{F}(Q_e^*) - (c-v) > 0$. Thus, $\partial\Omega_e^*/\partial Q_e^* = (p-v)\bar{F}(Q_e^*) - (c-v) > 0$. Also, we have $\partial\Omega_e^*/\partial c = -Q_e^*F(L_e(Q_e^*)) < 0$. So, when $c \in (v, \bar{c}_e)$, with $\partial\Omega_e^*/\partial w_e^* = 0$, $\partial\Omega_e^*/\partial Q_e^* > 0$,

$\partial\Omega_e^*/\partial c < 0$ and $dQ_e^*/dc \leq 0$, we have,

$$\frac{d\Omega_e^*}{dc} = \frac{\partial\Omega_e^*}{\partial w_e^*}\frac{dw_e^*}{dc} + \frac{\partial\Omega_e^*}{\partial Q_e^*}\frac{dQ_e^*}{dc} + \frac{\partial\Omega_e^*}{\partial c} < 0. \tag{A-4}$$

Q.E.D.

Proof of Corollary 2. Firstly, from the definitions of c^0, $\bar{c}_e$, $\bar{c}_b$, we have that $c^0 > \bar{c}_e$ and $c^0 > \bar{c}_b$. From Theorem 1, we know, if $c = c^0 > \bar{c}_e$, then $Q_e^* = \tilde{Q}$ and the total profit of the supply chain is $\Gamma_e^*(c^0) = (p-v)S(Q_e^*) - (c^0 - v)Q_e^* = (p-v)S(\tilde{Q}) - (c^0 - v)\tilde{Q}$. Obviously, when $c = c^0 = (p-v)\bar{F}(\tilde{Q}) + v$, we have $(p-v)\bar{F}(Q^0) = c^0 - v = (p-v)\bar{F}(\tilde{Q})$, and then $Q^0 = \tilde{Q}$. Thus, $\Gamma^0(c^0) = \Gamma_e^*(c^0) = (p-v)S(\tilde{Q}) - (c^0 - v)\tilde{Q}$. Q.E.D.

Proof of Theorem 2. (1) Firstly, we prove that, if $v < c \leq \dot{c}_e$, then $\Omega_e^* \geq \Omega_b^*$; otherwise, $\Omega_e^* < \Omega_b^*$.

With the result $\frac{d\Omega_b^*}{dc} = -\frac{S(Q_b^*)}{\bar{F}(Q_b^*)}$ in Corollary 1, we can show that $\frac{d^2(\Gamma(\tilde{Q})-\Omega_b^*)}{dc^2} = \frac{\bar{F}^2(Q_b^*)+S(Q_b^*)f(Q_b^*)}{\bar{F}^2(Q_b^*)} \cdot \frac{dQ_b^*}{dc} < 0$, which means that function $\Gamma(\tilde{Q}) - \Omega_b^*$ is a concave function in c. When $c = \bar{c}_b$, from the definition of $\bar{c}_b$, $(p-v)\bar{F}(\tilde{Q}) = (c-v)J(\tilde{Q})$. Thus, from Lemma 1, $\tilde{Q} = Q_b^*$. Then $\Gamma(\tilde{Q}) = \Gamma(Q_b^*) = \Omega_b^* + \Pi_b^* > \Omega_b^*$. So, when $c = \bar{c}_b$, there is $\Gamma(\tilde{Q}) - \Omega_b^* > 0$. From the definition of $\check{c}$, if $c = \check{c}$, then $\Gamma(\tilde{Q}) = 0$. Obviously, $\Omega_b^* = (p - w_b^*)S(Q_b^*) > 0$. So, when $c = \check{c}$, there is $\Gamma(\tilde{Q}) - \Omega_b^* < 0$. Thus, from the concavity of function $\Gamma(\tilde{Q}) - \Omega_b^*$, in the region of $(\bar{c}_b, \check{c})$, there exists a unique $\dot{c}_e$ satisfying $\Gamma(\tilde{Q}) - \Omega_b^* = 0$. Furthermore, As a result, we have $\Gamma(\tilde{Q}) > \Omega_b^*$ when $\bar{c}_b \leq c < \dot{c}_e$, and $\Gamma(\tilde{Q}) \leq \Omega_b^*$ when $c \geq \dot{c}_e$.

Now, we prove that $\dot{c}_e > \bar{c}_e$.

According to the definition of $\bar{c}_e$ and $\bar{c}_b$: $\bar{c}_e = \frac{(p-v)\bar{F}(\tilde{Q})}{1+2\bar{F}(\tilde{Q})} + v$, $\bar{c}_b = \frac{(p-v)\bar{F}(\tilde{Q})}{J(\tilde{Q})} + v$. Obviously, when $J(\tilde{Q}) \geq 1 + 2\bar{F}(\tilde{Q})$, we have $\bar{c}_b \leq \bar{c}_e$; otherwise, $\bar{c}_b > \bar{c}_e$.

Firstly, we consider the case $J(\tilde{Q}) \geq 1 + 2\bar{F}(\tilde{Q})$. So, in this case, there is $\bar{c}_b \leq \bar{c}_e$. When $c = \bar{c}_e$, there is $\frac{c-v}{p-v} = \frac{\bar{F}(\tilde{Q})}{1+2\bar{F}(\tilde{Q})} < \bar{F}(\tilde{Q})$. From the definition of Q^0, we have $\bar{F}(Q^0) = \frac{c-v}{p-v}$. Then, we have $\tilde{Q} < Q^0$. From the definition of $\bar{c}_b$, when $c = \bar{c}_b$, there is $Q_b^* = \tilde{Q}$. Furthermore, from $(p-v)\bar{F}(Q_b^*) = (c-v)J(Q_b^*)$, we know Q_b^* is decreasing in c. Since $\bar{c}_b \leq \bar{c}_e$, then, at point $c = \bar{c}_e$, we have $\tilde{Q} \geq Q_b^*$. Thus, when $c = \bar{c}_e$, $Q^0 > \tilde{Q} \geq Q_b^*$. From the definition of Q^0, for any given c, the Q^0 is the maximum point of $\Gamma(Q)$. So, when $c = \bar{c}_e$, we have $\Gamma(\tilde{Q}) \geq \Gamma(Q_b^*) > \Omega_b^*$, i.e., $\Gamma(\tilde{Q}) - \Omega_b^* > 0$. Thus, from the above definition of $\dot{c}_e$, we know that $\dot{c}_e$ must be greater than $\bar{c}_e$.

Secondly, we consider the case $J(\tilde{Q}) < 1 + 2\bar{F}(\tilde{Q})$. So, in this case, there is $\bar{c}_b > \bar{c}_e$. Since $\dot{c}_e > \bar{c}_b$, then we have $\dot{c}_e > \bar{c}_e$.

Consequently, the unique point $\dot{c}_e$ satisfies $\Gamma(\tilde{Q}) = \Omega_b^*$ and $\max\{\bar{c}_e, \bar{c}_b\} < \dot{c}_e < \check{c}$. Furthermore, from the concavity of function $\Gamma(\tilde{Q}) - \Omega_b^*$, we know: $\Gamma(\tilde{Q}) > \Omega_b^*$ when $\max\{\bar{c}_b, \bar{c}_e\} \leq c < \dot{c}_e$, and $\Gamma(\tilde{Q}) \leq \Omega_b^*$ when $c \geq \dot{c}_e$.

From Lemma 2 and Theorem 1, when $c \geq \bar{c}_e$, $w_e^* = c$, $Q_e^* = \tilde{Q}$. So, $\Omega_e^* = \Gamma(\tilde{Q})$. So, from the above results, we have: $\Omega_e^* > \Omega_b^*$ when $\max\{\bar{c}_b, \bar{c}_e\} \leq c < \dot{c}_e$, and $\Omega_e^* \leq \Omega_b^*$ when $c \geq \dot{c}_e$.

(2) Now, we show that $w_e^* \leq w_b^*$ and $\Pi_b^* \geq \Pi_e^*$ for any $c \in (v, p]$.

From Theorem 1, we know Q_e^* decreases with c. Then $L_e(Q_e^*)$ increases with c. Thus, $\frac{c-v}{w_e^*-v} = \frac{L_e(Q_e^*)}{Q_e^*}$ increases in c. From Lemma 1 and Corollary 1, we have that $(w_b^* - v)\bar{F}(Q_b^*) = c - v$, and Q_b^* decreases with c. Then $\frac{c-v}{w_b^*-v} = \bar{F}(Q_b^*)$ increases in c. When $c \to v$, $\frac{c-v}{w_e^*-v} \to 0$, $\frac{c-v}{w_b^*-v} \to 0$; When $c = \bar{c}_e < p$, $Q_e^* = L_e(Q_e^*) = \tilde{Q}$, $\bar{F}(Q_b^*) < 1$, and we have $\frac{c-v}{w_e^*-v} = 1 > \frac{c-v}{w_b^*-v}$. Thus, when $c < \bar{c}_e$, $\frac{c-v}{w_e^*-v} > \frac{c-v}{w_b^*-v}$, i.e., $w_e^* < w_b^*$. Furthermore, if $c > \bar{c}_e$, we have $\frac{c-v}{w_e^*-v} = 1$, and $\frac{c-v}{w_b^*-v} \leq 1$. Consequently, for all $c \in (v, p]$, we have $w_e^* \leq w_b^*$.

Next, we prove $\Pi_e^* \leq \Pi_b^*$. Recall that $\Pi_e^* = (w_e^* - v)[Y(Q_e^*) - Y(L_e(Q_e^*))]$ and $\Pi_b^* = (w_b^* - v)Y(Q_b^*)$. Given $c > \bar{c}_e$, $Q_e^* = L_e(Q_e^*)$ and $\Pi_e^* = 0$, $\Pi_b^* \geq 0$, then $\Pi_e^* \leq \Pi_b^*$. Right now, we consider the case of $c \in (v, \bar{c}_e]$. When $c = \bar{c}_e$, $Y(Q_e^*) - Y(L_e(Q_e^*)) = 0 < Y(Q_b^*)$. When $c \to v$, there is $Q_e^* \to +\infty$, $L_e(Q_e^*) \to 0$, $Q_b^* \to +\infty$, $Y(Q_e^*) - Y(L_e(Q_e^*)) = Y(Q_b^*) \to +\infty$. Since $Y(Q_e^*) - Y(L_e(Q_e^*))$ and $Y(Q_b^*)$ decrease in $c \in (v, \bar{c}_e]$, we get $Y(Q_e^*) - Y(L_e(Q_e^*)) < Y(Q_b^*)$. And $w_e^* < w_b^*$, we obtain $\Pi_e^* = (w_e^* - v)[Y(Q_e^*) - Y(L_e(Q_e^*))] < (w_b^* - v)Y(Q_b^*) = \Pi_b^*$. Thus, for any $c \in (v, p]$, $\Pi_e^* \leq \Pi_b^*$. Q.E.D.

Proof of Lemma 4. (1) In this part, we first prove there exists a unique point $w_e = w_e^M$ such that $\Pi_e(Q_e^*, w_e^M) = \Pi_b^*$. From Lemma 1, it is obvious that $p - v > (c - v)/\bar{F}(Q_b^*) = w_b^* - v$, i.e., $p > w_b^*$. For any given c, we have a constant $\Pi_b^*(c) \geq 0$. Then, when $w_e = c$, we have $\Pi_e(Q_e^*, c) = 0 \leq \Pi_b^*$; when $w_e = p$, from (3) and (2) we have,

$$\begin{aligned}
\Pi_e(Q_e^*, p) &= \mathbb{E}\left[(p - v)\min[D, Q_e^*(p)] - (c - v)Q_e^*(p)\right]^+ \geq \mathbb{E}\left[(p - v)\min[D, Q_b^*] - (c - v)Q_b^*\right]^+ \\
&\geq \mathbb{E}\left[(w_b^* - v)\min[D, Q_b^*] - ((1 + r_b^*)c - v)Q_b^*\right]^+ = \Pi_b^*.
\end{aligned}$$

The above first inequality results from that the $Q_e^*(p)$ is the optimal decision. From Lemma 3, we know $\Pi_e(Q_e^*, w_e)$ increases in w_e. Thus, for a given c, we have a unique $w_e = w_e^M$ satisfying $\Pi_e(Q_e^*, w_e^M) = \Pi_b^*$.

(2) Now, we prove, for any $c \in (v, p]$, $w_e^M(c) \leq w_b^*(c)$ by contradiction approach.

Firstly, we suppose, for some $c \in (v, p]$ there is $w_e^M(c) > w_b^*(c)$. Then, we have,

$$\begin{aligned}
&\Pi_b^* = \mathbb{E}\left[(w_b^*(c) - v)\min[D, Q_b^*] - ((1 + r_b^*)c - v)Q_b^*\right]^+ \\
< \; &\mathbb{E}\left[(w_e^M(c) - v)\min[D, Q_b^*] - ((1 + r_b^*)c - v)Q_b^*\right]^+ \leq \mathbb{E}\left[(w_e^M(c) - v)\min[D, Q_b^*] - (c - v)Q_b^*\right]^+ \\
\leq \; &\mathbb{E}\left[(w_e^M(c) - v)\min[D, Q_e^*(w_e^M(c))] - (c - v)Q_e^*(w_e^M(c))\right]^+ = \Pi_e(Q_e^*, w_e^M(c)).
\end{aligned}$$

The above last inequality results from that the $Q_e^*(w_e^M(c))$ is the optimal decision when $w_e = w_e^M(c)$. The result $\Pi_b^* < \Pi_e(Q_e^*, w_e^M(c))$ is contradictory with the definition of $w_e^M(c)$. Thus, for any given c, we have $w_e^M(c) \leq w_b^*(c)$.

From Theorem 2, we have $\Pi_e^*(c) < \Pi_b^*(c)$. So, for $c \in (v, p]$, there is $\Pi_e^*(Q_e^*, w_e^*(c)) = \Pi_e^*(c) < \Pi_e(Q_e^*, w_e^M(c))$. From Lemma 3, for any given c, we have that Π_e^* increases in w_e. Thus, we obtain $w_e^*(c) < w_e^M(c)$. So, we obtain $w_e^*(c) < w_e^M(c) \le w_b^*(c)$. Q.E.D.

Proof of Lemma 5. (1) From Lemma 2, we know Q_e^* increases in w_e. When $w_e = p$, the manufacturer sets an optimal production quantity $Q_e^*(p)$, and the retailer gets a profit $\Omega_e^*(p) = \Omega_e(Q_e^*(p), p)$. From Eq. (5), for any given c, when $w_e = p$, $\Omega_e^*(p) = \mathbb{E}\min[(p-v)\min[D, Q_e^*(p)], (c-v)Q_e^*(p)] - (c-v)Q_e^*(p) \le 0 < \Omega_b^*$. Thus, we have the result: $\Omega_e^*(p) < \Omega_b^*$ for any $c \in (v, \dot{c}_e]$.

(2) We prove there exists a unique $w_e^R \in (w_e^*, p)$ satisfying $\Omega_e(Q_e^*, w_e^R) = \Omega_b^*$ when $\bar{c}_b \le c < \dot{c}_e$.

From the proof of Theorem 2, we get $\Gamma(\tilde{Q}) > \Omega_b^*$. Also, for a given $c \in [\bar{c}_b, \dot{c}_e)$, when $w_e = c$, we have $\Gamma(\tilde{Q}) = \Gamma_e(Q_e^*, c) = \Omega_e(Q_e^*, c) + \Pi_e(Q_e^*, c) = \Omega_e(Q_e^*, c)$. Then, we have, $\Omega_e(Q_e^*, c) > \Omega_b^*$.

In the following, we consider two cases: $\bar{c}_b \le \bar{c}_e$ (i.e., $J(\tilde{Q}) \ge 1 + 2\bar{F}(\tilde{Q})$) and $\bar{c}_b > \bar{c}_e$ (i.e., $J(\tilde{Q}) < 1 + 2\bar{F}(\tilde{Q})$).

Case 1: $\bar{c}_b \le \bar{c}_e$**.** There are two sceneries: $\bar{c}_b \le c \le \bar{c}_e$ and $\bar{c}_e < c < \dot{c}_e$.

When $\bar{c}_e < c \le \dot{c}_e$, from the proof of Theorem 1, for all w_e, we have $d\Omega_e(Q_e^*, w_e)/dw_e < 0$. We already know, when $w_e = c$, $\Omega_e(Q_e^*, c) > \Omega_b^*$, and from part (1), when $w_e = p$, $\Omega_e^*(p) < \Omega_b^*$. Furthermore, from Lemma 3, $\Omega_e(Q_e^*, w_e)$ is a unimodal function of w_e. Thus, there exists a unique w_e^R, which satisfies $\Omega_e(Q_e^*, w_e^R) = \Omega_b^*$. Since $w_e^* = c$, then $w_e^R > w_e^*$.

Now, we consider the case $\bar{c}_b \le c \le \bar{c}_e$. From the proof of Theorem 1, when $w_e = w_e^*$, $\Omega_e(Q_e^*, w_e)$ reaches the maximum value. Thus $\Omega_e(Q_e^*, w_e^*) > \Omega_e(Q_e^*, c) > \Omega_b^*$. Thus, with the result $\Omega_e^*(p) < \Omega_b^*$ in part (1) and unimodal property of $\Omega_e(Q_e^*, w_e)$, we know that there exists a unique $w_e^R \in (w_e^*, p]$, which satisfies $\Omega_e(Q_e^*, w_e^R) = \Omega_b^*$.

Case 2: $\bar{c}_b > \bar{c}_e$**.** We consider the case $\bar{c}_b \le c < \dot{c}_e$. In this case, from the proof of Theorem 1, when $w_e = w_e^*$, we know $d\Omega_e(Q_e^*, w_e)/dw_e < 0$. Similarly with the above scenario $\bar{c}_e < c \le \dot{c}_e$ in case 1, we can find a w_e^R satisfying $\Omega_e(Q_e^*, w_e^R) = \Omega_b^*$, and $w_e^R > w_e^* = c$.

(3) We prove there exists a unique $w_e^R \in (w_e^*, p)$ satisfying $\Omega_e(Q_e^*, w_e^R) = \Omega_b^*$ when $v \le c < \bar{c}_b$.

Since $v \le c < \bar{c}_b \le \max\{\bar{c}_e, \bar{c}_b\} \le \dot{c}_e$, then from Theorem 2, we have $\Omega_e(Q_e^*, w_e^*) = \Omega_e^* > \Omega_b^*$. With the result of $\Omega_e^*(p) < \Omega_b^*$ in part (1), and the property that $\Omega_e(Q_e^*, w_e)$ is a unimodal function, in the region (w_e^*, p), there exists a unique w_e^R, which satisfies $\Omega_e(Q_e^*, w_e^R) = \Omega_b^*$ and $w_e^R > w_e^*$.

Thus, from parts (2) and (3), we know there exists a unique $w_e^R \in (w_e^*, p)$ satisfying $\Omega_e(Q_e^*, w_e^R) = \Omega_b^*$ for any $v \le c < \dot{c}_e$. Q.E.D.

Proof of Theorem 3. 1. From Lemma 5, when $c = \dot{c}_e$, $w_e^R(c) = c$. From Lemma 4, for any given $c \in (v, p]$, we have $w_e^M(c) > c$. Thus, there must exist a point $\hat{c}_e < \dot{c}_e$ such that $w_e^M(\hat{c}_e) = w_e^R(\hat{c}_e)$.

2. From the above result, for any fixed $c \in (v, \dot{c}_e]$, there may exist two scenarios between w_e^M

and w_e^R. When $c \geq \hat{c}_e$, we have $w_e^M > w_e^R$, the unique sub-game perfect financing equilibrium is bank financing. When $c < \hat{c}_e$, we have $w_e^M < w_e^R$, i.e., there exists a Pareto zone $w_e \in [w_e^M, w_e^R]$ such that $\Pi_e(Q_e^*, w_e) \geq \Pi_b^*$ and $\Omega_e(Q_e^*, w_e) \geq \Omega_b^*$. Thus, the unique sub-game perfect financing equilibrium is early payment financing.

3. From above results, we know, when $v < c < \hat{c}_e$, the unique financing equilibrium is early payment financing; when $\hat{c}_e \leq c \leq p$, the equilibrium is bank financing. Q.E.D.

Proof of Theorem 4. Following the same procedure in Theorem 3, we can easily get the results. Q.E.D.

Proof of Lemma 6. Firstly, following the same procedure in Lemma 3, we can easily prove $\Pi_I(Q_I^*, w_I)$ increases with w_I

Now, we prove that, for any given w_I, $\Pi_I(Q_I^*, w_I)$ decreases with r_I.

With the equations of $(w_I - v)\bar{F}(Q_I^*) = (c(1+r_I) - v)\bar{F}(L_I(Q_I^*))$ and $L_I(Q_I^*) = \frac{(c(1+r_I)-v)Q_I^*}{w_I - v}$, we can show that, $-(w_I - v)f(Q_I^*)\frac{dQ_I^*}{dr_I} = c\bar{F}(L_I(Q_I^*)) - (c(1+r_I) - v)f(L_I(Q_I^*))\frac{dL_I(Q_I^*)}{dr_I}$, $\frac{dL_I(Q_I^*)}{dr_I} = \frac{cQ_I^*}{w_I - v} + \frac{c(1+r_I)-v}{w_I - v}\frac{dQ_I^*}{dr_I}$. So, we have, $\frac{dQ_I^*}{dr_I} = \frac{cQ_I^*(M(Q_I^*)-1)}{(1+r_I)c-v}$, $\frac{dL_I(Q_I^*)}{dr_I} = \frac{cQ_I^*M(Q_I^*)}{w_I - v}$. Thus,

$$dQ_I^*/dr_I \leq 0, dL_I(Q_I^*)/dr_I \geq 0. \tag{A-5}$$

So, for any r_I, there is

$$\frac{d\Pi_I(Q_I^*, w_I)}{dr_I} = (w_I - v)\left[Q_I^* f(Q_I^*)\frac{dQ_I^*}{dr_I} - L_I(Q_I^*)f(L_I(Q_I^*))\frac{dL_I(Q_I^*)}{dr_I}\right] \leq 0. \tag{A-6}$$

So, for a fixed purchasing price w_I, the manufacturer's payoffs function $\Pi_I(Q_I^*, w_I)$ decreases with r_I.

(1) First, we prove that, for any $c \in (v, \hat{c}_I]$, the point $w_I^M(r_I)$ is increasing in r_I.

Similarly with Lemma 2, the profit function of the manufacturer $\Pi_I(Q_I, w_I)$ is a unimodal function in Q_I. From Eq. (6), for any given c and w_I, $\Pi_I(Q_I, w_I)$ is decreasing in r_I. Thus, for any given c and w_I, the maximum value of the manufacturer's profit function, $\Pi_I(Q_I^*, w_I)$, must decrease in r_I. Furthermore, we know $\Pi_I(Q_I^*, w_I)$ increases in w_I. From the definition of w_I^M, we have $\Pi_I(Q_I^*, w_I^M) = \Pi_b^*$. Thus, the point w_I^M must increase in r_I.

Now, we prove that, for any $c \in (v, \hat{c}_I]$, $Q_I^*(w_I^M(r_I), r_I)$ decreases with r_I.

Consider r_I^1 and r_I^2 with $0 \leq r_I^1 < r_I^2 \leq \hat{r}_I$. With the result that w_I^M increases in r_I, we have $w_I^M(r_I^1) < w_I^M(r_I^2)$. From the above result, for any r_I^i, $w_I^M(r_I^i)$, $i = 1, 2$, we have $V(Q_I^*(w_I^M(r_I^i), r_I^i)) = V(L(Q_I^*(w_I^M(r_I^i), r_I^i)))$. Then, from Eq. (6), we have

$$\Pi_I(Q_I^*(w_I^M(r_I^i), r_I^i), w_I^M(r_I^i)) = (w_I^M(r_I^i) - v)\int_{L(Q_I^*(w_I^M(r_I^i), r_I^i))}^{Q_I^*(w_I^M(r_I^i), r_I^i)} xf(x)dx.$$

With the definition of $w_I^M(r_I^i)$, the value of above functions must equal Π_b^*. Since $w_I^M(r_I^1) < w_I^M(r_I^2)$, we must have $Q_I^*(w_I^M(r_I^1), r_I^1) > Q_I^*(w_I^M(r_I^2), r_I^2)$. Thus, $Q_I^*(w_I^M(r_I), r_I)$ decreases in r_I.

(2) From Theorem 2, the $\dot{c}_e$ satisfies $\Gamma(\tilde{Q}) = \Omega_b^*$. Also the $\dot{c}_I$ satisfies $\Gamma(\tilde{Q}) = \Omega_b^*$. Thus, we

have $\dot{c}_e = \dot{c}_I$.

(3) Now, we prove that $\hat{c}_I > \hat{c}_e$.

(i) Firstly, we show that, when $c = \hat{c}_e$, $Q_e^*(w_e^M) > Q_I^*(w_I^M) > Q_b^*$.

According to the definitions of w_e^M and w_e^R, we have $\Pi_e(Q_e^*, w_e^M) = \Pi_b(Q_b^*)$ and $\Omega_e(Q_e^*, w_e^R) = \Omega_b(Q_b^*)$. Since, when $c = \hat{c}_e$, we have $w_e^M = w_e^R$. So, $\Gamma_e(Q_e^*, w_e^M) = \Gamma_b(Q_b^*)$. We know the supply chain total profit $\Gamma(Q)$ is a concave function of Q, and Q^0 is the optimal point. So we must have $Q_e^*(w_e^M) > Q^0 > Q_b^*$.

Also, according to the definitions of w_I^M, we have $\Pi_I(Q_I^*, w_I^M) = \Pi_b(Q_b^*)$. From Theorem 1, and $c = \hat{c}_e > \bar{c}_e$, the retailer's optimal decision is $w_I^* = \hat{c}_e(1+r_I)$. Since $c = \hat{c}_e > \bar{c}_b$, from Lemma 7, the manufacturer's optimal decision $Q_I^*(w_I^*) > Q_b^*$. Follow the same procedure in Lemma 2, we can show that $Q_I^*(w_I)$ increases with w_I. Since $w_I^M > w_I^*$, then, $Q_I^*(w_I^M) > Q_I^*(w_I^*)$. Thus, $Q_I^*(w_I^M) > Q_b^*$. From the above result in part (1), with the fixed $c = \hat{c}_e$, we already have $Q_e^*(w_e^M) > Q_I^*(w_I^M)$. Thus, we have $Q_e^*(w_e^M) > Q_I^*(w_I^M) > Q_b^*$.

(ii) Now, we prove that $\hat{c}_I > \hat{c}_e$.

Since the supply chain's profit function $\Gamma(Q)$ is concave on Q, when $c = \hat{c}_e$, with the results $\Gamma_e(Q_e^*, w_e^M) = \Gamma_b(Q_b^*)$ and $Q_e^*(w_e^M) > Q_I^*(w_I^M) > Q_b^*$, we have $\Gamma_I(Q_I^*, w_I^M) > \Gamma_e(Q_e^*, w_e^M)$. We already have $\Omega_e(Q_e^*, w_e^M) = \Omega_b(Q_b^*)$, $\Pi_I(Q_I^*, w_I^M) = \Pi_e(Q_e^*, w_e^M) = \Pi_b(Q_b^*)$. Thus, $\Omega_I(Q_I^*, w_I^M) = \Gamma_I(Q_I^*, w_I^M) - \Pi_I(Q_I^*, w_I^M) = \Gamma_I(Q_I^*, w_I^M) - \Pi_e(Q_e^*, w_e^M) > \Gamma_e(Q_e^*, w_e^M) - \Pi_e(Q_e^*, w_e^M) = \Omega_e(Q_e^*, w_e^M) = \Omega_b(Q_b^*)$. Thus, we get, $\Omega_I(Q_I^*, w_I^M) > \Omega_b(Q_b^*)$, $\Pi_I(Q_I^*, w_I^M) = \Pi_b(Q_b^*)$. So, when $c = \hat{c}_e$, we have $w_I^R > w_I^M$. Thus, we get $\hat{c}_I > \hat{c}_e$. Q.E.D.

Proof of Theorem 5. Recall Corollary 2. Given $r_I = 0$, the early payment financing coordinates the supply chain when $c = c^0$. Then we have $Q^0(c) = \tilde{Q}$ if $c = c^0$. Both $Q^0(c)$ and $Q_I^*(w_I^*)$ decrease in c, and $Q_I^*(w_I^*) \geq \tilde{Q}$. Then we have $Q_I^*(w_I^*) < Q^0(c)$ if $c < c^0$; otherwise, $Q_I^*(w_I^*) = \tilde{Q} > Q^0(c)$.

From Lemma 6, we know $Q_I^*(w_I^M(r_I), r_I)$ decreases with r_I. Thus, with the definition of $\tilde{c}_3$, $\tilde{c}_2$ and $\tilde{c}_1$, we have $\tilde{c}_3 < \tilde{c}_2 < \tilde{c}_1 < c^0$.

For any given $c \in (v, \hat{c}_I]$, $r_I \in [0, \hat{r}_I]$ and $w_I = w_I^M(r_I)$, the retailer's profit function is: $\Omega_I(Q_I^*, w_I^M(r_I)) = (p-v)S(Q_I^*(w_I^M(r_I))) - (c-v)Q_I^*(w_I^M(r_I)) - \Pi_b^*$. And the supply chain total profit $pS(Q) - cQ$ is concave, where the maximum point is at the point of $Q^0(c)$. So, if $Q_I^*(w_I^M(r_I)) < Q^0(c)$, $\Omega_I(Q_I^*, w_I^M(r_I))$ increases in Q_I^*; otherwise, $\Omega_I(Q_I^*, w_I^M(r_I))$ decreases in Q_I^*.

If $c < \tilde{c}_3$, $Q_I^*(w_I^M(r_I)) < Q_I^*(w_I^M(0)) < Q^0(c)$. And we have that $\Omega_I(Q_I^*, w_I^M(r_I))$ increases in Q_I^*. Since $Q_I^*(w_I^M(r_I))$ decreases in r_I (from Lemma 6), $\Omega_I(Q_I^*, w_I^M(r_I))$ decreases in r_I, then we have $\Omega_e(Q_e^*, w_e^M) = \Omega_I(Q_I^*, w_I^M(0)) > \Omega_I(Q_I^*, w_I^M(r_I))$.

If $c > \tilde{c}_1$, $Q_I^*(w_I^M(0)) > Q_I^*(w_I^M(\hat{r}_I)) > Q^0(c)$. And $\Omega_I(Q_I^*, w_I^M(r_I))$ decreases in Q_I^*. Since

$Q_I^*(w_I^M(r_I))$ decreases in r_I, $\Omega_I(Q_I^*, w_I^M(r_I))$ monotonically increases in r_I.

If $c < \tilde{c}_1$, we have $Q_I^*(w_I^M(\hat{r}_I)) < Q^0(c)$. Since $Q_I^*(w_I^M(r_I))$ decreases in r_I, we can find an $r_I^* < \hat{r}_I$ solving $Q_I^*(w_I^M(r_I^*)) = Q^0(c)$. Recall $\tilde{c}_2$ solving $\Omega_I(Q_I^*, w_I^M(\hat{r}_I)) = \Omega_I(Q_I^*, w_I^M(0))$. We have $Q_I^*(w_I^M, r_I^*(\tilde{c}_2)) = Q^0(\tilde{c}_2)$. Given $c = \tilde{c}_2$, $\Omega_I^M(Q_I^*, w_I^M(r_I), \tilde{c}_2)$ decreases in $r_I \in [r_I^M(\tilde{c}_2), \hat{r}_I]$. Then $\tilde{c}_2 \leq c < \tilde{c}_1$, $\Omega_I^M(Q_I^*, w_I^M(r_I)) > \Omega_I(Q_I^*, w_I^M(0))$, and $\Omega_I^M(Q_I^*, w_I^M(r_I))$ firstly increases but then decreases in r_I.

If $\tilde{c}_3 \leq c < \tilde{c}_2$, we can find an $r_I^M < \hat{r}_I$ solving $\Omega_I^M(Q_I^*, w_I^M(r_I^M)) = \Omega_I(Q_I^*, w_I^M(0))$. For any given $c \in [\tilde{c}_3, \tilde{c}_2)$, we have $\Omega_I^M(Q_I^*, w_I^M(r_I)) \geq \Omega_I(Q_I^*, w_I^M(0))$. If $r_I \leq r_I^M(c)$, $\Omega_I^M(Q_I^*, w_I^M(r_I)) < \Omega_I(Q_I^*, w_I^M(0))$. Q.E.D.

Proof of Theorem 6. Part 1. We prove that for any $c \in (v, p]$, $\hat{r}_I(c)$ decreases in c.

From the proof of Lemma 6, for any given c, at $w_I = p$, $\Pi_I(Q_I^*(c), p, r_I)$ decreases in r_I. Follow the same procedure in the proof of Theorem 1, we can easily prove that, for any given r_I, $dQ_I^*/dc < 0$, and $dL_I(Q_I^*(c))/dc > 0$. We know the manufacturer's optimal profit is $\Pi_I(Q_I^*(c), p, r_I) = (p - v)[Y(Q_I^*(c)) - Y(L_I(Q_I^*(c)))]$. Thus, for any given r_I, $\Pi_I(Q_I^*(c), p, r_I)$ decreases in c.

Now, we consider two cases $d\Pi_b^*(c)/dc \geq 0$ and $d\Pi_b^*(c)/dc < 0$.

(i) Suppose $d\Pi_b^*(c)/dc \geq 0$. Let $\grave{c}_1 < c < \grave{c}_2$, we have $\Pi_b^*(\grave{c}_1) \leq \Pi_b^*(\grave{c}_2)$. We already know, for any feasible r_I, $\Pi_I(Q_I^*(\grave{c}_1), p, r_I) > \Pi_I(Q_I^*(\grave{c}_2), p, r_I)$. With the definition of $\hat{r}_I(c)$, $\Pi_I(Q_I^*(\grave{c}_1), p, \hat{r}_I(\grave{c}_1)) = \Pi_b^*(\grave{c}_1)$, $\Pi_I(Q_I^*(\grave{c}_2), p, \hat{r}_I(\grave{c}_2)) = \Pi_b^*(\grave{c}_2)$. Since $d\Pi_b^*(c)/dc \geq 0$ and $\grave{c}_1 < \grave{c}_2$, we have $\Pi_I(Q_I^*(\grave{c}_2), p, \hat{r}_I(\grave{c}_2)) = \Pi_b^*(\grave{c}_2) > \Pi_b^*(\grave{c}_1) = \Pi_I(Q_I^*(\grave{c}_1), p, \hat{r}_I(\grave{c}_1)) > \Pi_I(Q_I^*(\grave{c}_2), p, \hat{r}_I(\grave{c}_1))$. Thus, we know $\hat{r}_I(\grave{c}_1) > \hat{r}_I(\grave{c}_2)$, i.e., $d\hat{r}_I(c)/dc < 0$.

(ii) Suppose $d\Pi_b^*(c)/dc < 0$. Let $\grave{c}_1 < c < \grave{c}_2$, we have $\Pi_b^*(\grave{c}_1) > \Pi_b^*(\grave{c}_1)$. Thus $\Pi_I(Q_I^*(\grave{c}_1), p, \hat{r}_I(\grave{c}_1)) > \Pi_I(Q_I^*(\grave{c}_2), p, \hat{r}_I(\grave{c}_2))$. With the relation $\Pi_I(Q_I^*(c), p, r_I) = (p-v)[Y(Q_I^*(c)) - Y(L_I(Q_I^*(c)))]$. Thus, we have $Q_I^*(\grave{c}_1) > Q_I^*(\grave{c}_2)$, and then $L_I(Q_I^*(\grave{c}_1), \hat{r}_I(\grave{c}_1)) < L_I(Q_I^*(\grave{c}_2), \hat{r}_I(\grave{c}_2))$. So, $\frac{(1+\hat{r}_I(\grave{c}_1))\grave{c}_1 - v}{p-v} = \frac{L_I(Q_I^*(\grave{c}_1), \hat{r}_I(\grave{c}_1))}{Q_I^*(\grave{c}_1)} < \frac{L_I(Q_I^*(\grave{c}_2), \hat{r}_I(\grave{c}_2))}{Q_I^*(\grave{c}_2)} = \frac{(1+\hat{r}_I(\grave{c}_2))\grave{c}_2 - v}{p-v}$. Thus, we know $(1 + \hat{r}_I(c))c$ increases in c and $(1 + \hat{r}_I(c))c \leq p$. When $c = p$, we know $(1 + \hat{r}_I(p))p \leq p$, and we must have $\hat{r}_I(p) = 0$. So, we must also have $d\hat{r}_I(c)/dc < 0$. Therefore, for any $c \in (v, p]$, $\hat{r}_I(c)$ decreases in c.

Part 2. Now, we consider the optimal interest rate r_I^* in $[\tilde{c}_3, \hat{c}_I]$.

For any given $\tilde{c}_3 < c \leq \tilde{c}_1$, the optimal optimal rate r_I^* satisfies $(w_I^M(r_I^*) - v)\bar{F}(Q^0(c)) = (c(1 + r_I^*) - v)\bar{F}(L_I(Q^0(c)))$, $\Pi_I^*(w_I^M(r_I^*), Q^0(c)) = \Pi_b^*$, and $Q^0(c)$ solves $\bar{F}(Q^0(c)) = \frac{c-v}{p-v}$. We have $\frac{L_I(Q^0(c))}{Q^0(c)} = \frac{c(1+r_I^*(c))-v}{w_I^M(c)-v}$. If c increases, we have $\frac{L_I(Q^0(c))}{Q^0(c)}$ increases in c. And then $\frac{c(1+r_I^*(c))-v}{w_I^M(c)-v}$ increases in c. Let $\tilde{c}_0$ solves $c(1+r_I^*(c)) = w_I^M(c)$, then we obtain $Q^0(\tilde{c}_0) = \tilde{Q}$. If $c \in [\tilde{c}_3, \tilde{c}_1]$ increases, $\frac{c(1+r_I^*(c))-v}{w_I^M(c)-v} < 1$ increases to 1. Since $w_I^M(c)$ increases in c, to keep $\frac{c(1+r_I^*(c))-v}{w_I^M(c)-v}$ increasing to reach c, we must have $r_I^*(c)$ increase in c. Recall that $(w_I^M(\hat{r}_I) - v)\bar{F}(Q^0(c)) = (c(1 + \hat{r}_I) - v)\bar{F}(L_I(Q^0(c)))$, and we have $r_I^*(\tilde{c}_1) = \hat{r}_I$. As a result, for any given $c \in [\tilde{c}_3, \tilde{c}_1)$, $r_I^*(c) \leq \hat{r}_I(\tilde{c}_1)$. When $c \in [\tilde{c}_1, \hat{c}_I]$, the optimal interest rate reaches the upper bound (i.e., $r_I^* = \hat{r}_I$), and we already know $\hat{r}_I$ decreases with c. Q.E.D.

Proof of Theorem 7. The results are directly from Theorems 5 and 6, and are thus skipped for brevity. Q.E.D.

Proof of Theorem 8. We first study the firms' preferences between early payment and bank financing supposing both are available in the following lemma.

Lemma 8 *Consider the case with an identical wholesale price w_x in bank financing and early payment financing (i.e., $w_b = w_e = w_x$).*

1. *For any $c \in (v, p]$ and $w_x \in [c, p]$, the manufacturer always prefers early payment to bank financing (i.e., $\Pi_b(Q_b^*, w_x) < \Pi_e(Q_e^*, w_x)$);*
2. *If $v < c \leq \check{c}$, then there exists a unique $w_x^E \in (w_e^*, p)$ such that, when $w_x \in [c, w_x^E]$, the retailer prefers early payment (i.e., $\Omega_b(Q_b^*, w_x) \leq \Omega_e(Q_e^*, w_x)$); when $w_x \in (w_x^E, p]$, the retailer prefers bank financing (i.e., $\Omega_b(Q_b^*, w_x) > \Omega_e(Q_e^*, w_x)$). If $\check{c} < c \leq p$, then for any $w_x \in [c, p]$, the retailer prefers bank financing (i.e., $\Omega_b(Q_b^*, w_x) > \Omega_e(Q_e^*, w_x)$).*

Lemma 8 with an identical wholesale price is parallel to Theorem 2 with non-identical wholesale prices. Lemma 8 first indicates that the manufacturer always prefers early payment to bank financing if the wholesale price is identical, which is different from when the wholesale prices are not identical. The underlying reason is that when the wholesale price is not identical, the retailer will command a big enough wholesale price discount in early payment. But, when the wholesale prices are fixed at the same level in both financing schemes, which are assumed to be available in Lemma 8, it is intuitive that the manufacturer will always prefer early payment because of the benefit of paying no interest.

For the retailer, the preference structure is similar to that without identical wholesale prices, although the outcome also depends on the identical wholesale price level. Overall, when the production cost is high enough, the retailer still prefers the manufacturer to use bank financing due to the manufacturer's overwhelming default risk. This demonstrates that the production cost continues to play an important role in the retailer's equilibrium financing decision, and the retailer's preference between the two financing schemes does not qualitatively alter in terms of whether the wholesale price is identical.

We now list the firms' preferences between in-house factoring and bank financing supposing both are available as follows.

Lemma 9 *Consider the case with an identical wholesale price w_x in bank financing and in-house factoring financing (i.e., $w_b = w_I = w_x$).*

1. *For any $c \in (v, p]$ and $w_x \in [c, p]$, there exists an interest rate r_x^M, such that: when $r_I \in [0, r_x^M)$, the manufacturer prefers in-house factoring (i.e., $\Pi_b(Q_b^*, w_x) < \Pi_I(Q_I^*, w_x, r_I)$); when $r_I \geq r_x^M$, the manufacturer prefers bank financing (i.e., $\Pi_b(Q_b^*, w_x) \geq \Pi_I(Q_I^*, w_x, r_I)$).*
2. *If $v < c \leq \check{c}$, then there exists a unique w_x^I, where $w_x^I > w_x^E$, such that, when $w_x \in [c, w_x^I]$ and $r_I \in [0, r_x^M]$, the retailer prefers in-house factoring (i.e., $\Omega_b(Q_b^*, w_x) \leq \Omega_I(Q_I^*, w_x)$); when $w_x \in (w_x^I, p]$, the retailer prefers bank financing (i.e., $\Omega_b(Q_b^*, w_x) > \Omega_I(Q_I^*, w_x)$). If $\check{c} < c \leq p$, then for any $w_x \in [c, p]$, the retailer prefers bank financing (i.e., $\Omega_b(Q_b^*, w_x) > \Omega_I(Q_I^*, w_x)$).*

Lemma 9 with an identical wholesale price is comparable to Theorem 4 with non-identical wholesale prices. This comparison is similar to that between Lemma 8 and Theorem 2 and the underlying rationale is similar. Again, the retailer's preference to in-house factoring is the same as that in Theorem 4 when the production cost is big enough.

Based on Lemmas 8 and 9, the results of Theorem 8 are therefore straightforward. Q.E.D.

Proof of Lemma 8. We prove the two items sequently. 1. Firstly, from (2) and (3), when the wholesale price in bank financing and early payment are the same at w_x, we have $\Pi_b(Q, w_x) = E[w_x \min[D, Q] + v(Q - D)^+ - c(1 + r_b^*)Q]^+$, $\Pi_e(Q, w_x) = E[w_x \min[D, Q] + v(Q - D)^+ - cQ]^+$. Because $r_b^* > 0$, for any Q, there is $c(1 + r_b^*)Q > cQ$, and then $\Pi_b(Q, w_x) < \Pi_e(Q, w_x)$. Specially, there is $\Pi_b(Q_b^*, w_x) < \Pi_e(Q_b^*, w_x)$. Secondly, since Q_e^* is the maximum point of $\Pi_e(Q_e, w_x)$, for any Q_e there is $\Pi_e(Q_e, w_x) \leq \Pi_e(Q_e^*, w_x)$. Specially, there is $\Pi_e(Q_b^*, w_x) \leq \Pi_e(Q_e^*, w_x)$. Therefore, we have $\Pi_b(Q_b^*, w_x) < \Pi_e(Q_e^*, w_x)$.

2. Firstly, we consider the properties of $\Omega_b(Q_b^*, w_x)$. From $(w_x - v)\bar{F}(Q_b^*) = c - v$, we have $\frac{dQ_b^*}{dw_x} = \frac{1}{(w_x - v)^2 f(Q_b^*)} \geq 0$. Thus, with $\Omega_b(Q_b^*, w_x) = (p - w_x)S(Q_b^*)$, we have $\frac{d\Omega_b(Q_b^*, w_x)}{dw_x} = \frac{p - w_x}{(w_x - v)^2 h(Q_b^*)} - S(Q_b^*)$. Obviously, $p - w_x$, $\frac{1}{(w_x - v)^2}$, $\frac{1}{h(Q_b^*)}$ and $-S(Q_b^*)$ are decreasing with w_x. Thus, $\frac{d\Omega_b(Q_b^*, w_x)}{dw_x}$ is decreasing with w_x, and then $\Omega_b(Q_b^*, w_x)$ is a unimodel function of w_x. When $w_x = c$, $Q_b^* = 0$, and then $S(Q_b^*) = 0$ and $\Omega_b(Q_b^*, c) = (p - w_x)S(Q_b^*) = 0$. When $w_x = p$, $\Omega_b(Q_b^*, p) = (p - w_x)S(Q_b^*) = 0$.

Secondly, we consider the relation of $\Omega_e(Q_e^*, w_x)$ and $\Omega_b(Q_b^*, w_x)$ at the point of $w_x = p$. From (5), when $w_x = p$, $\Omega_e(Q_e^*, p) = -(c - v)Q_e^* + E \min[(p - v)\min[D, Q_e^*], (c - v)Q_e^*] \leq 0$. We already know the property of $\Omega_b(Q_b^*, p) = 0$. Thus, at the point of $w_x = p$, $\Omega_e(Q_e^*, p) \leq \Omega_b(Q_b^*, p)$.

Lastly, we analyze the threshold w_x^E. There are three cases: $c \in (v, \dot{c}_e]$, $c \in [\dot{c}_e, \check{c}]$ and $c \in [\check{c}, p]$.

(i) In the case $c \in (v, \dot{c}_e]$. From the Proof of Theorem 1, for $w_x \geq w_e^*$, $\Omega_e(Q_e^*, w_x)$ is decreasing with w_x. Furthermore, from Theorem 2, $\Omega_e(Q_e^*, w_e^*) \geq \Omega_b(Q_b^*, w_b^*)$. Because $\Omega_b(Q_b^*, w_x)$ is a unimodel function of w_x, for any w_x, $\Omega_b(Q_b^*, w_b^*) \geq \Omega_b(Q_b^*, w_x)$. Specially, there is $\Omega_b(Q_b^*, w_b^*) \geq \Omega_b(Q_b^*, w_e^*)$. Therefore, at the point of $w_x = w_e^*$, $\Omega_e(Q_e^*, w_e^*) \geq \Omega_b(Q_b^*, w_e^*)$. We already know $\Omega_e(Q_e^*, p) \leq \Omega_b(Q_b^*, p)$. So, there must exist a unique $w_x^R \in (w_e^*, p]$ such that $\Omega_b(Q_b^*, w_x^R) = \Omega_e(Q_e^*, w_x^R)$. Thus, when $w_x \in [c, w_x^R]$, there is $\Omega_b(Q_b^*, w_x) \leq \Omega_e(Q_e^*, w_x)$, when $w_x \in (w_x^E, p]$, there is $\Omega_b(Q_b^*, w_x) > \Omega_e(Q_e^*, w_x)$.

(ii) In the case $c \in (\dot{c}_e, \check{c}]$. We first consider the properties of $\Omega_e(Q_e^*, w_x)$. From Lemma 3 and the proof of Theorem 1, when $c \in (\dot{c}_e, \check{c}] \in [\bar{c}_e, p]$, $\Omega_e(Q_e^*, w_x)$ is decreasing in w_x and the maximum point is c. Furthermore, from Theorem 1, in the case of $c \in (\dot{c}_e, \check{c}]$, we have $w_e^* = c$ and $Q_e^* = L_e(Q_e^*) = \tilde{Q}$. Thus, from (4), $\Pi_e^* = 0$, and then $\Omega_e(Q_e^*, w_e^*) = \Gamma(\tilde{Q}) = (p-v)S(\tilde{Q}) - (c-v)\tilde{Q}$. From the definition of $\check{c}$, if $c \leq \check{c}$, then $(p-v)S(\tilde{Q}) - (c-v)\tilde{Q} > 0$. Thus, at the point of $c \in (\dot{c}_e, \check{c}]$, there is $\Omega_e(Q_e^*, c) \geq 0$. We already know $\Omega_b(Q_b^*, c) = 0$. So, $\Omega_e(Q_e^*, c) \geq \Omega_b(Q_b^*, c)$. We already know $\Omega_e(Q_e^*, p) \leq \Omega_b(Q_b^*, p)$. So, there must exist a unique $w_x^E \in (c, p]$ such that $\Omega_b(Q_b^*, w_x^E) = \Omega_e(Q_e^*, w_x^E)$. Thus, when $w_x \in [c, w_x^E]$, there is $\Omega_b(Q_b^*, w_x) \leq \Omega_e(Q_e^*, w_x)$, when $w_x \in (w_x^E, p]$, there is $\Omega_b(Q_b^*, w_x) > \Omega_e(Q_e^*, w_x)$.

(iii) In the case of $c \in (\check{c}, p]$. From Theorem 1, when $c \geq \check{c} > \bar{c}_e$, there is $w_e^* = c$ and $Q_e^* = L_e(Q_e^*) = \tilde{Q}$. Thus, from (4), $\Pi_e^* = 0$, and then $\Omega_e(Q_e^*, w_e^*) = \Gamma(\tilde{Q}) = (p-v)S(\tilde{Q}) - (c-v)\tilde{Q}$. From the definition of $\check{c}$, if $c > \check{c}$, then $(p-v)S(\tilde{Q}) - (c-v)\tilde{Q} < 0$. Thus, given a $c \in (\check{c}, p]$ there is $\Omega_e(Q_e^*, c) < 0$. Since c is the maximum point of $\Omega_e(Q_e^*, w_x)$. So for any w_x, there is $\Omega_e(Q_e^*, w_x) \leq \Omega_e(Q_e^*, c) < 0$. Furthermore, for any w_x, $\Omega_b(Q_b^*, w_x) \geq 0$. Thus, for any $w_x \in [c, p]$, there is $\Omega_b(Q_b^*, w_x) \geq \Omega_e(Q_e^*, w_x)$. Q.E.D.

Proof of Lemma 9. We prove the three items sequently. 1. Obviously, when $r_I = 0$, there is $Q_I^* = Q_e^*$, and $\Pi_I(Q_I^*, w_x, 0) = \Pi_e(Q_e^*, w_x)$. Furthermore, from Lemma 8(1), $\Pi_b(Q_b^*, w_x) < \Pi_e(Q_e^*, w_x)$. So, $\Pi_b(Q_b^*, w_x) \leq \Pi_I(Q_I^*, w_x, 0)$. Obviously, when $r_I \to +\infty$, $\Pi_I(Q_I^*, w_x, r_I) \to 0$. From (A-6), we have $d\Pi_I(Q_I^*, w_x, r_I)/dr_I \leq 0$. Thus, there must exist an interest rate r_x^M, such that: $\Pi_b(Q_b^*, w_x) = \Pi_I(Q_I^*, w_x, r_x^M)$, when $r_I \in [0, r_x^M)$, $\Pi_b(Q_b^*, w_x) < \Pi_I(Q_I^*, w_x, r_I)$; when $r_I \geq r_x^M$, $\Pi_b(Q_b^*, w_x) \geq \Pi_I(Q_I^*, w_x, r_I)$.

2. Follow the same proof procedure in Lemma 3(2), we can prove that, for a fixed r_I, the retailer's profit $\Omega_I(Q_I^*, w_x)$ is a unimodal function on w_x. So, with the same proof procedure in Lemma 8(2), there must exist an $r_I \geq 0$, such that there exists a unique w_x^I such that, when $w_x \in [c, w_x^I]$, we have $\Omega_b(Q_b^*, w_x) \leq \Omega_I(Q_I^*, w_x)$; when $w_x \in (w_x^I, p]$, we have $\Omega_b(Q_b^*, w_x) > \Omega_I(Q_I^*, w_x)$.

Now, we need to prove that, when $r_I \in [0, r_x^M]$, there is $w_x^I > w_x^E$.

From (8), we know,

$$\frac{d\Omega_I(Q_I^*, w_x, r_I)}{dr_I} = \frac{cw_x Q_I^* \bar{F}(Q_I^*)}{(1+r_I)c - v} U(Q_I^*), \tag{A-7}$$

where

$$U(Q_I^*) = 1 - (1 - M(Q_I^*)) \frac{(p-v)\bar{F}(Q_I^*) - (c-v)}{(w_x - v)\bar{F}(Q_I^*)}. \tag{A-8}$$

From the definition of $U(Q_e^*)$ in (A-3), when $r_I = 0$, $Q_e^* = Q_I^*$, and then $U(Q_I^*) = U(Q_e^*)$. From (A-1), $\frac{dQ_e^*}{dw_e} > 0$. So, from Lemma 10, we know $1 - M(Q_e^*)$ decreases in w_e. Obviously, $\frac{(p-v)\bar{F}(Q_e^*) - (c-v)}{\bar{F}(Q_e^*)} = (p-v) - \frac{c-v}{\bar{F}(Q_e^*)}$ decreases in w_e. Also, $\frac{1}{w_e - v}$ decreases in w_e. Thus, $(1 -$

$M(Q_e^*))\frac{(p-v)\bar{F}(Q_e^*)-(c-v)}{(w_x-v)\bar{F}(Q_e^*)}$ decreases in w_e. So, $U(Q_e^*)$ must increase in w_e. Following this same procedure, we know $U(Q_I^*)$ is also increasing in w_e. From (A-5), $\frac{dQ_I^*}{dr_I} < 0$. So, from Lemma 10, we know $M(Q_I^*)$ decreases in r_I. So, $1 - M(Q_I^*)$ increases in r_I. Also $\bar{F}(Q_I^*)$ increases in r_I. So, $\frac{(p-v)\bar{F}(Q_I^*)-(c-v)}{\bar{F}(Q_I^*)} = (p-v) - \frac{c-v}{\bar{F}(Q_I^*)}$ increases in r_I. Thus, for a fixed w_x, $(1-M(Q_I^*))\frac{(p-v)\bar{F}(Q_I^*)-(c-v)}{(w_x-v)\bar{F}(Q_I^*)}$ increases in r_I. Then, for a fixed w_x, $U(Q_I^*)$ decreases in r_I. So, for a fixed w_x, $U(Q_e^*) \geq U(Q_I^*)$.

We then show $w_x^I > w_x^E$ in two cases: $\tilde{c}^0 < c \leq \check{c}$ and $v < c \leq \tilde{c}^0$ as the following, where $\tilde{c}^0$ satisfies $Q^0 = \tilde{Q}$.

(i) $\tilde{c}^0 < c \leq \check{c}$. From the definition of $\tilde{c}^0$, when $\tilde{c}^0 < c$, there is $(p-v)\bar{F}(\tilde{Q}) - (c-v) < 0$. Since $Q_I^* \geq \tilde{Q}$, then $(p-v)\bar{F}(Q_I^*) - (c-v) < 0$. So, from (A-8), $U(Q_I^*) > 0$. So, from (A-7), $\frac{d\Omega_I(Q_I^*, w_x, r_I)}{dr_I} > 0$, which means, for fixed w_x and r_I, $\Omega_I(Q_I^*, w_x, r_I)$ is always greater than $\Omega_e(Q_e^*, w_x)$. Furthermore, with the definitions of w_x^E, there is $\Omega_b(Q_b^*, w_x^E) = \Omega_e(Q_e^* w_x^E) < \Omega_I(Q_I^*, w_x^E)$. From the definitions of w_x^I, we have $\Omega_b(Q_b^*, w_x^I) = \Omega_I(Q_I^*, w_x^I)$. Since $\Omega_I(Q_I^*, w_x)$ is decreasing in w_x when $w_x \geq w_I^*$, then there must have $w_x^E < w_x^I$.

(ii) Case $v < c \leq \tilde{c}^0$. From the proof of Theorem 1, $K(Q_e^*) = -V(Q_e^*)U(Q_e^*) - (Y(Q_e^*) - Y(L(Q_e^*)))$. And, when $w_x = w_e^*$, there is $K(Q_e^*) = 0$. Furthermore, we have $Y(Q_e^*) - Y(L(Q_e^*)) > 0$ and $V(Q_e^*) > 0$. So, when $w_x = w_e^*$, there is $U(Q_e^*) < 0$. We already know, for a fixed w_x, $U(Q_e^*) \geq U(Q_I^*)$. Thus, when $w_x = w_e^*$, there is $U(Q_I^*) < 0$. Now, consider the point $w_e = p$. Since $0 \leq 1 - M(Q_I^*) \leq 1/2$ and $\frac{(p-v)\bar{F}(Q_I^*)-(c-v)}{(p-v)\bar{F}(Q_I^*)} < 1$, then $(1 - M(Q_I^*))\frac{(p-v)\bar{F}(Q_I^*)-(c-v)}{(p-v)\bar{F}(Q_I^*)} \leq 1/2$, and then $U(Q_I^*) > 1/2$. We already know $U(Q_I^*)$ is increasing in w_e. Thus, there must exist one point $\grave{w}_x \in (w_e^*, p)$, such that: when $w_e \in (v, \grave{w}_x)$, $U(Q_I^*) \leq 0$; when $w_e \in (\grave{w}_x, p]$, $U(Q_I^*) > 0$. Then, from (A-7), when $w_e \in (v, \grave{w}_x)$, $\frac{d\Omega_I(Q_I^*, w_x, r_I)}{dr_I} < 0$; when $w_e \in (\grave{w}_x, p]$, $\frac{d\Omega_I(Q_I^*, w_x, r_I)}{dr_I} > 0$. With the same prove procedure, there must have $w_x^E < w_x^I$.

3. When $c \geq \tilde{c}^0$, $Q_I^* \geq \tilde{Q} \geq Q^0$. Thus, $\check{c} > \tilde{c}^0$. So, when $\check{c} < c \leq p$, $\Omega_I(Q_I^*, w_x)$ is increasing in r_I. When r_I equals the maximum value $w/c - 1$, $Q_I^* = \tilde{Q}$, and then $\Omega_I(Q_I^*, w_x) = \Gamma_I(Q_I^*, w_x) - \Pi_I(Q_I^*, w_x) = pS(\tilde{Q}) - c \leq 0$. So, for any r_I, $\Omega_I(Q_I^*, w_x) \leq \Omega_b(Q_b^*, w_x)$. Thus, for any $w_x \in [c, p]$, we have $\Omega_b(Q_b^*, w_x) > \Omega_I(Q_I^*, w_x)$. Q.E.D.

Proof of Lemma 7. From Lemma 1, obviously, Q_b^* decreases in c. Then, if $c \leq \bar{c}_b$, we have $Q_b^* \geq \tilde{Q}$; otherwise, $Q_b^* < \tilde{Q}$. From Theorem 1, we have Q_e^* decreases in $c \in (v, \bar{c}_e)$ and $Q_e^* = \tilde{Q}$ if $c \geq \bar{c}_e$. Thus, with the above results, we know, when $\bar{c}_b \leq \bar{c}_e$ and $c \in (\bar{c}_b, p]$, we have $Q_e^* > Q_b^*$.

We next prove that $Q_e^* > Q_b^*$ if $c < \bar{c}_e$. We use the contradiction approach. Suppose $Q_e^* < Q_b^*$ in $c \in (v, \bar{c}_e)$. As we know $Q_e^* = Q_b^*$ if $c = v$, and $Q_e^* > Q_b^*$ if $c = \bar{c}_e$. Since both Q_e^* and Q_b^* decrease in c, there exists a $c < \bar{c}_e$ such that $Q_b^* = Q_e^*$. According to Lemma 1 and Lemma 2, given $c > v$, we have $Q_b^* \neq Q_e^*$. It is contradictory. Then we obtain that $Q_e^* > Q_b^*$ in $c \in (v, \bar{c}_e)$.

Similarly, we can prove that, when $\bar{c}_b > \bar{c}_e$, $Q_e^* \le Q_b^*$ if $c < \bar{c}_b$; and $Q_e^* > Q_b^*$ when $c \in (\bar{c}_b, p]$. Q.E.D.

Proof of Theorem 9. Firstly, we have $\Gamma(Q) = (p-v)S(Q) - (c-v)Q$, and $\frac{d\Gamma(Q)}{dc} = ((p-v)\bar{F}(Q) - (c-v))\frac{dQ}{dc} - Q$.

(1) In the case of $J(\tilde{Q}) \ge 1 + 2\bar{F}(\tilde{Q})$, from Lemma 7, we have $\bar{c}_b \le \bar{c}_e$. We consider three scenarios of c as follows.

(i) Case $c \in (v, \bar{c}_b]$. From Theorem 1, we know $dQ_e^*/dc < 0$. Since $c < \bar{c}_e$, we know, $(p - v)\bar{F}(Q_e^*) - (c-v) > 0$. Then, we have $Q_e^* < Q^0$ and $d\Gamma_e^*(c)/dc = ((p-v)\bar{F}(Q_e^*) - (c-v))dQ_e^*/dc - Q_e^* < 0$. When $c \to v$, $\Gamma_e^*(c) = \Gamma_b^*(c)$. When $c = \bar{c}_b$, we know $Q_b^* = \tilde{Q}$, and then $\Gamma_b^*(c) = \Gamma(\tilde{Q})$. Thus, when $c = \bar{c}_b$, $\Gamma_e^*(c) > \Gamma(\tilde{Q}) = \Gamma_b^*(c)$. Since $\Gamma_e^*(c)$ and $\Gamma_b^*(c)$ decrease in c when $c \in (v, \bar{c}_b]$, then $\Gamma_e^*(c) > \Gamma_b^*(c)$.

(ii) Case $c \in (\bar{c}_b, \bar{c}_e)$. From Lemma 7, we know $Q_e^* > Q_b^*$. We already know $(p-v)\bar{F}(Q_e^*) - (c - v) > 0$. Then, we have $Q^0 > Q_e^* > Q_b^*$, and then $\Gamma_e^*(c) \ge \Gamma_b^*(c)$.

(iii) Case $c \in (\bar{c}_e, p]$. From Theorem 1, $Q_e^* = \tilde{Q}$. We already know $Q_b^* < \tilde{Q}$. Thus $Q_e^* > Q_b^*$. Also, we have $d\Gamma_e^*(c)/dc = ((p-v)\bar{F}(Q_e^*) - (c-v))dQ_e^*/dc - Q_e^* = -Q_e^* < 0$.

As we already know, when $c = \bar{c}_e$, we have $\Gamma_e^*(c) > \Gamma_b^*(c)$. Obviously, when $c = p$, $Q_b^* = 0$ and $Q_e^* = \tilde{Q}$, then we have $\Gamma_b^*(c) = 0$ and $\Gamma_e^*(c) < 0$. Consequently, there exists a unique point $c = \ddot{c}_e$ solving $\Gamma_e^*(c) = \Gamma_b^*(c)$. Thus, we have: if $c \in (\bar{c}_e, \ddot{c}_e]$, then $\Gamma_e^*(c) \ge \Gamma_b^*(c)$; if $c \in (\ddot{c}_e, p]$, $\Gamma_e^*(c) < \Gamma_b^*(c)$.

Combining the above results, we have: if $c \in (v, \ddot{c}_e]$, then $\Gamma_e^*(c) \ge \Gamma_b^*(c)$; if $c \in (\ddot{c}_e, p]$, $\Gamma_e^*(c) < \Gamma_b^*(c)$.

(2) In the case of $J(\tilde{Q}) < 1 + 2\bar{F}(\tilde{Q})$, from Lemma 7, there is $\bar{c}_b > \bar{c}_e$. We also inspect three scenarios of c as follows.

(i) Case $c \in (v, \bar{c}_e]$. As in (1)-(i), we know $dQ_e^*/dc < 0$ and $(p-v)\bar{F}(Q_e^*) - (c-v) > 0$. Then, we have $d\Gamma_e^*(c)/dc < 0$. When $c \to v$, $\Gamma_e^*(c) = \Gamma_b^*(c)$. When $c = \bar{c}_e$, $\Gamma_e^*(c) < \Gamma_b^*(c)$. Since $\Gamma_e^*(c)$ and $\Gamma_b^*(c)$ decrease in c when $c \in (v, \bar{c}_e]$, then $\Gamma_e^*(c) < \Gamma_b^*(c)$.

(ii) Case $c \in (\bar{c}_e, \bar{c}_b]$. From Lemma 7, we know $Q_e^*(c) \le Q_b^*(c)$. Obviously, $Q_b^*(c) < Q^0(c)$. So, $\Gamma_e^*(c) \le \Gamma_b^*(c)$. At the point $c = \bar{c}_b$, we know $Q_b^*(c) = \tilde{Q} = Q_e^*(c)$, and then $\Gamma_e^*(c) = \Gamma_b^*(c)$.

(iii) Case $c \in (\bar{c}_b, p]$. As the same as (1)-(iii), we have $Q_e^*(c) > Q_b^*(c)$, $d\Gamma_e^*(c)/dc < 0$. So, there exists a unique $c = \ddot{c}_e$ solving $\Gamma_e^*(c) = \Gamma_b^*(c)$ and we have: if $c \in (\bar{c}_b, \ddot{c}_e]$, then $\Gamma_e^*(c) \ge \Gamma_b^*(c)$; if $c \in (\ddot{c}_e, p]$, then $\Gamma_e^*(c) < \Gamma_b^*(c)$.

Combining the above results, we have: if $c \in (v, \bar{c}_b)$, then $\Gamma_e^* < \Gamma_b^*$; if $c \in [\bar{c}_b, \ddot{c}_e]$, then $\Gamma_e^* \ge \Gamma_b^*$; if $c \in (\ddot{c}_e, p]$, $\Gamma_e^* < \Gamma_b^*$.

At last, we prove $\dot{c}_e > \ddot{c}_e$ by contradiction. Suppose $\dot{c}_e \le \ddot{c}_e$, we get $\Gamma_e^*(\dot{c}_e) \ge \Gamma_b^*(\dot{c}_e)$. Then, we have, $\Omega_e^*(\dot{c}_e) + \Pi_e^*(\dot{c}_e) \geq \Omega_b^*(\dot{c}_e) + \Pi_b^*(\dot{c}_e)$. Since $\Omega_e^*(\dot{c}_e) = \Omega_b^*(\dot{c}_e)$, we have $\Pi_e^*(\dot{c}_e) \geq \Pi_b^*(\dot{c}_e)$.

Since $\dot{c}_e \geq \max\{\bar{c}_e, \bar{c}_b\}$, $\Pi_e^*(\dot{c}_e) = 0$ and $\dot{c}_e < p$, we have $\Pi_b^*(\dot{c}_e) > 0$. Then, we have $\Pi_e^*(\dot{c}_e) < \Pi_b^*(\dot{c}_e)$. This contradicts the above result. Therefore, we have $\dot{c}_e > \ddot{c}_e$. Q.E.D.

Lemma 10 *We denote* $M(Q_e^*) = \frac{H(Q_e^*)-1}{H(Q_e^*)-H(L_e(Q_e^*))}$, *where* Q_e^* *satisfies* $(w_e - v)\bar{F}(Q_e^*) = (c - v)\bar{F}(L_e(Q_e^*))$. *The properties of* $M(Q_e^*)$ *are as follows,*

1. $M(Q_e^*)$ *increases with* w_e *in* $(c, p]$;
2. $1/2 \leq M(Q_e^*) \leq 1$, *and* $M(Q_e^*) = 1/2$ *when* $w_e = c$.

Proof of Lemma 10. We prove the two items sequentially. (1) From the Proof of Lemma 2, in the region of $c < w_e \leq p$, we have $L_e(Q_e^*) < \tilde{Q} < Q_e^*$. With the $h(Q)$ increasing property stated in the Model section, we can prove that $H(Q)$ is also increasing in Q and, thus, $H(L_e(Q_e^*)) \leq H(\tilde{Q}) \leq H(Q_e^*)$. From the definition of $\tilde{Q}$, we have $H(\tilde{Q}) = 1$. Then, $H(L_e(Q_e^*)) \leq 1 \leq H(Q_e^*)$. Thus, for all $c < w_e \leq p$, from the definition of $M(Q_e^*)$ in Lemma 10, we have $M(Q_e^*) \leq 1$. Then,

$$\frac{\partial M(Q_e^*)}{\partial Q_e} = \frac{H'(Q_e^*)(1 - H(L_e(Q_e^*))) + H'(L_e(Q_e^*))(H(Q_e^*) - 1)\frac{c-v}{w_e-v}}{(H(Q_e^*) - H(L_e(Q_e^*)))^2} > 0. \tag{A-9}$$

Because $Q_e^*(w_e)$ is a function of w_e, we can rewrite $L_e(Q_e^*) = \frac{c-v}{w_e-v}Q_e^*(w_e)$. So, $\frac{\partial L_e(Q_e^*)}{\partial w_e} = -\frac{L_e(Q_e^*)}{w_e-v}$ and

$$\frac{dL_e(Q_e^*)}{dw_e} = (c - v)\frac{(w_e - v)dQ_e^*/dw_e - Q_e^*}{(w_e - v)^2} = -\frac{(c - v)Q_e^* M(Q_e^*)}{(w_e - v)^2}. \tag{A-10}$$

Then,

$$\begin{aligned}
\frac{dM(Q_e^*)}{dw_e} &= \frac{\partial M(Q_e^*)}{\partial Q_e} \cdot \frac{dQ_e^*}{dw_e} + \frac{\partial M(Q_e^*)}{\partial w_e} \\
&= \frac{H'(Q_e^*)(1 - H(L_e(Q_e^*))) + H'(L_e(Q_e^*))(H(Q_e^*) - 1)\frac{c-v}{w_e-v}}{(H(Q_e^*) - H(L_e(Q_e^*)))^2} \cdot \frac{Q_e^*(1 - M(Q_e^*))}{w_e - v} \\
&\quad - \frac{H'(L_e(Q_e^*))(H(Q_e^*) - 1)\frac{L_e(Q_e^*)}{w_e-v}}{(H(Q_e^*) - H(L_e(Q_e^*)))^2} \\
&= \frac{Q_e^*(1 - H(L_e(Q_e^*)))^2 \Theta(w_e)}{(w_e - v)(H(Q_e^*) - H(L_e(Q_e^*)))^3},
\end{aligned} \tag{A-11}$$

where $\Theta(w_e) = H'(Q_e^*) - H'(L_e(Q_e^*))\Upsilon(w_e)$, and $\Upsilon(w_e) = \frac{L_e(Q_e^*)}{Q_e^*} \cdot \frac{(H(Q_e^*)-1)^2}{(1-H(L_e(Q_e^*)))^2}$.

Taking derivative of $\Upsilon(w_e)$ with respect to w_e, we have,

$$\begin{aligned}
\frac{d\Upsilon(w_e)}{dw_e} &= \frac{1}{Q_e^*(1 - H(L_e(Q_e^*)))^2}\left[\frac{dL_e(Q_e^*)}{dw_e}(H(Q_e^*) - 1)^2 + 2L_e(Q_e^*)(H(Q_e^*) - 1)H'(Q_e^*)\frac{dQ_e^*}{dw_e}\right] \\
&\quad - \frac{L_e(Q_e^*)(H(Q_e^*) - 1)^2}{(Q_e^*)^2(1 - H(L_e(Q_e^*)))^4}\left[\frac{dQ_e^*}{dw_e}(1 - H(L_e(Q_e^*)))^2 - 2Q_e^*(1 - H(L_e(Q_e^*)))H'(L_e(Q_e^*))\frac{dL_e(Q_e^*)}{dw_e}\right] \\
&= \frac{L_e(Q_e^*)(H(Q_e^*) - 1)\left[-(H(Q_e^*) - 1) + 2Q_e^*(1 - M(Q_e^*))\Theta(w_e)\right]}{(w_e - v)Q_e^*(1 - H(L_e(Q_e^*)))^2}.
\end{aligned} \tag{A-12}$$

Because Q_e^* satisfies $(w_e - v)\bar{F}(Q_e^*) = (c - v)\bar{F}(L_e(Q_e^*))$, then given $L_e(Q_e^*) = \frac{c-v}{w_e-v}Q_e^*(w_e)$ and $V(Q) = Q\bar{F}(Q)$, we have $V(Q_e^*) = V(L_e(Q_e^*))$. Obviously, when $w_e = c$, we have $Q_e^* = L_e(Q_e^*) = \tilde{Q}$, and then we have $\Upsilon(w_e) = 1$ and $\Theta(w_e) = 0$.

To prove $\frac{dM(Q_e^*)}{dw_e} > 0$ in $(c, p]$, according to (A-11) and $H(Q_e^*) > H(L_e(Q_e^*))$, we need to show $\Theta(w_e) > 0$, which is related to $\Upsilon(w_e)$. Because the region $(c, p]$ can be separated into intervals that have a value of either $\frac{d\Upsilon(w_e)}{dw_e} \geq 0$ or $\frac{d\Upsilon(w_e)}{dw_e} < 0$, we consider the following two scenarios of $\frac{d\Upsilon(w_e)}{dw_e}$ in any interval $(\acute{w}_e^1, \acute{w}_e^2]$, which is a subdomain inside $(c, p]$ having a value of either $\frac{d\Upsilon(w_e)}{dw_e} \geq 0$ or $\frac{d\Upsilon(w_e)}{dw_e} < 0$ but not both.

Scenario 1: Suppose$\frac{d\Upsilon(w_e)}{dw_e} \geq 0$ in $(\acute{w}_e^1, \acute{w}_e^2]$. From the above assumption and (A-12), we obtain

$$\frac{d\Upsilon(w_e)}{dw_e} = \frac{L_e(Q_e^*)(H(Q_e^*) - 1)\left[-(H(Q_e^*) - 1) + 2Q_e^*(1 - M(Q_e^*))\Theta(w_e)\right]}{(w_e - v)Q_e^*(1 - H(L_e(Q_e^*)))^2} \geq 0.$$

Because $H(Q_e^*) > 1$, there must be $-(H(Q_e^*)-1)+2Q_e^*(1-M(Q_e^*))\Theta(w_e) \geq 0$. So, from $H(Q_e^*) > 1$ and $M(Q_e^*) < 1$, we have $\Theta(w_e) \geq \frac{H(Q_e^*)-1}{2Q_e^*(1-M(Q_e^*))} > 0$.

Scenario 2: Suppose $\frac{d\Upsilon(w_e)}{dw_e} < 0$ in $(\acute{w}_e^1, \acute{w}_e^2]$. Given that $h(Q)$ is increasing and convex in Q, we know $h'(Q) > 0$ and $h''(Q) > 0$. Then $H'(Q) = h(Q) + Qh'(Q) > 0$ and $H''(Q) = 2h'(Q) + Qh''(Q) > 0$ (i.e., both $H(Q)$ and $H'(Q)$ are increasing in Q). So, based on that Q_e^* is increasing in w_e (from (A-1)) and $L_e(Q_e^*)$ is decreasing in w_e (from (A-10)), we know $H'(Q_e^*)$ is increasing and $H'(L_e(Q_e^*))$ is decreasing in w_e. Thus, with the condition $\frac{d\Upsilon(w_e)}{dw_e} < 0$ in this scenario, the function $\Theta(w_e) = H'(Q_e^*) - H'(L_e(Q_e^*))\Upsilon(w_e)$ must increase with w_e. Therefore, we prove that $\Theta(w_e) \geq 0$ in the scenario of $\frac{d\Upsilon(w_e)}{dw_e} < 0$ in $(\acute{w}_e^1, \acute{w}_e^2]$.

Given that the interval $(\acute{w}_e^1, \acute{w}_e^2]$ is any subdomain inside the feasible domain of $(c, p]$, we therefore prove that for all w_e in $(c, p]$, there is $\Theta(w_e) > 0$. Then, from (A-11), with the results of $\Theta(w_e) > 0$ and $H(Q_e^*) > H(L_e(Q_e^*))$, we have $\frac{dM(Q_e^*)}{dw_e} > 0$ (i.e., $M(Q_e^*)$ increases with w_e).

(2) Denote $\bar{M} = \lim_{w_e \to c} M(Q_e^*)$. Since $\lim_{w_e \to c} Q_e^* = \tilde{Q}$ and $\lim_{w_e \to c} L_e(Q_e^*) = \tilde{Q}$, the following relations hold: $\lim_{w_e \to c} \frac{dQ_e^*}{dw_e} = \frac{\tilde{Q}}{c-v}(1 - \bar{M})$, $\lim_{w_e \to c} \frac{dL_e(Q_e^*)}{dw_e} = -\frac{\tilde{Q}\bar{M}}{c-v}$, $\lim_{w_e \to c} H'(Q_e^*) = \lim_{w_e \to c} H'(L_e(Q_e^*)) = H'(\tilde{Q})$. Then, with L Hospital rule, we have

$$\bar{M} = \lim_{w_e \to c} \frac{H'(Q_e^*)dQ_e^*/dw_e}{H'(Q_e^*)dQ_e^*/dw_e - H'(L_e(Q_e^*))dL_e(Q_e^*)/dw_e} = \frac{\frac{\tilde{Q}}{c-v}(1 - \bar{M})}{\frac{\tilde{Q}}{c-v}(1 - \bar{M}) + \frac{\tilde{Q}}{c-v}\bar{M}} = 1 - \bar{M}.$$

Thus, $\bar{M} = 1/2$, i.e., $M(Q_e^*) = 1/2$, when $w_e = c$. Q.E.D.