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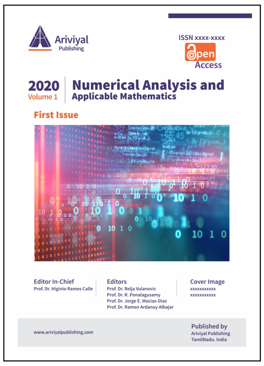

Using the Kellogg-Tsan Solution Decomposition in Numerical Methods for Singularly Perturbed Convection-Diffusion Problems

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Abstract: The linear one-dimensional singularly perturbed convection-diffusion problem is solved numerically by a second-order method that is uniform in the perturbation parameter ε. The method uses the Kellogg-Tsan decomposition of the continuous solution. This increases the accuracy of the numerical results and simplifies the proof of their ε-uniformity.

Keywords: singular perturbation; convection-diffusion; Kellogg-Tsan solution decomposition; Vulanović-Bakhvalov mesh; finite differences; uniform convergence.

1. Introduction

In their seminal work, Kellogg and Tsan[9] introduced a special decomposition of the solution to the following singularly perturbed convection-diffusion problem,

$$Lu := -\varepsilon u'' - b(x)u' + c(x)u = f(x),\ x \in I := [0,1],\ u(0) = u(1) = 0, \tag{1}$$

with a small perturbation parameter ε, $0 < \varepsilon \ll 1$, and sufficiently smooth functions $b, c, f \in C^k(I)$, where k is a non-negative integer and, moreover, b and c satisfy

$$b(x) > \beta > 0,\ \ c(x) \geq 0\ \text{ for }\ x \in I. \tag{2}$$

Under these conditions, the problem has a unique solution $u \in C^{k+2}(I)$, which generally shows an exponential boundary layer near $x = 0$. For $x \in I$, the Kellogg-Tsan decomposition is

$$u(x) = v(x) + z(x),\ \ v(x) = -\frac{\varepsilon u'(0)}{b(0)} e^{-b(0)x/\varepsilon}, \tag{3}$$

where the component z is such that

$$|z^{(j)}(x)| \leq C\left(1 + \varepsilon^{1-j} e^{-\beta x/\varepsilon}\right),\ \ j = 0, 1, \ldots, k+1. \tag{4}$$

The function v is the boundary-layer component, which is not known because the value of $u'(0)$ is not available. Nevertheless, v is given by an explicit formula and this is why Kellogg and Tsan were able to use their decomposition to prove ε-uniform convergence of the exponentially fitted Il'in scheme[8] on the uniform mesh. The explicitness of the boundary-layer component separates the Kellogg-Tsan decomposition from other solution decompositions. Of those, the Shishkin decomposition (see [13, p. 59] and [4, p. 46] for instance) is undoubtedly the most well-known one (see also the version in [11, p. 62], which relaxes some of the conditions required). These other decompositions produce a regular component, which is smoother than z in (4) w.r.t. the dependence on ε, and a singular (boundary-layer) component, which behaves like v, but is not given explicitly, rather, its derivative estimates are proved.

The problem (1)–(2) is the most frequently considered model problem of singular-perturbation type. Singularly perturbed differential equations are of interest because they occur in various applications.[3, 13, 4] Since their solutions change abruptly in narrow intervals called boundary or interior layers, singular perturbation problems require special numerical methods that produce errors which are uniform in ε. The above-mentioned Il'in scheme is one of such methods belonging to the class of schemes that are fitted to the behavior of the solution. Another class of methods uses discretization meshes like Bakhvalov's[2] and Shishkin's.[17, 13] In the present paper, we consider one of the methods of the latter kind, a finite-difference discretization of the problem (1) on a Bakhvalov-type mesh.

There is a striking difference between convection-diffusion problems of type (1)–(2) and reaction-diffusion problems which can be described as (1) with $b \equiv 0$ and $c > 0$ on I. For the latter, the consistency (truncation) error of finite-difference schemes on a layer-adapted mesh is uniform in ε, but this is not the case for (1)–(2).[22] This means that one only needs an ε-uniformly stable discretization on a layer-adapted mesh to get ε-uniform convergence for reaction-diffusion problems, whereas convection-diffusion problems require special techniques for proving ε-uniform convergence. One such technique uses hybrid stability inequalities based on the discrete Green's function.[6, 1, 12, 11] Another one is the technique that uses barrier functions to analyze the consistency error.[13, 4, 11, 15] Relatively recently, the third method of proof has been proposed.[22] In this paper, the discrete equations in the layer are scaled so that the consistency error becomes uniform in ε, but, as a trade off, the proof of ε-uniform stability gets more complicated. At the same time, the scaling makes the discrete system well-conditioned w.r.t. ε,[16] which is why the technique is referred to as the *preconditioning technique*. A more recent example of the application of this technique is[14].

However, there is yet another, not so well-known, method, introduced in[20], for achieving and proving ε-uniform convergence for the problem (1)–(2). The method is based on the Kellogg-Tsan decomposition (3). The idea of [20] is to consider an auxiliary problem related to (1),

$$Lu = f(x), \quad x \in I, \quad -\varepsilon u'(0) = \gamma, \quad u(1) = 0, \tag{5}$$

where γ is a constant independent of ε. The auxiliary problem (5) has a unique solution $u_\gamma \in C^{k+2}(I)$. The purpose of the subscript γ is dual: to indicate that the solution of (5) depends on γ (the need for this becomes obvious in the discussion that follows, as well as in Section 4) and to distinguish this solution from u, the solution of (1). The solution u_γ has the Kellogg-Tsan decomposition $u_\gamma = v_\gamma + z_\gamma$ with the components respectively corresponding to v and z in (3). The change of the Dirichlet boundary condition at $x = 0$ into a Neumann one renders the boundary-layer function v_γ completely known. Then, u_γ is replaced with $v_\gamma + z_\gamma$ in the problem (5) and a numerical approximation of z_γ is found by some ε-uniform method. The proof of ε-uniform convergence is relatively easy because the consistency error for z_γ is uniform in ε. At the same time, we can expect more accurate results than when (5) is solved directly for u_γ by the same numerical method because the v_γ-component of the solution does not contribute to the error at all. Of course, the question is how to get the solution u of the problem (1) from u_γ. In fact, two problems of type (5) should be solved with different values of γ, γ_1 and γ_2. Then, u can be expressed as a linear combination of u_{γ_1} and u_{γ_2}.

The main purpose of the present paper is to make the idea from [20] more widely known. At the same time, we improve the numerical method used in [20]. The finite-difference scheme used there is a second-order four-point hybrid scheme. We show that the simpler three-point combination of the standard central scheme and the midpoint upwind scheme,[18, 14] produces an accuracy of the same order. We also correct the scheme for discretizing the Neumann boundary condition at $x = 0$. Like in, [20], we use one of the Vulanović's modifications of the Bakhvalov mesh[19] to discretize the problem.

The outline of the paper is as follows. In Section 2, we present the discretization mesh and the finite-difference schemes that are used in our numerical method. Section 3 is on the numerical method for solving the auxiliary problem (5). After that, we show in Section 4 how a numerical approximation of the solution to the problem (1) can be constructed from two numerical solutions of problems of type (5). In Section 5, we present the results of numerical experiments. Finally, we provide some concluding remarks in Section 6.

2. The Discretization Mesh and Finite-Difference Schemes

Let I^N be the discretization mesh with points x_i, $i = 0, 1, \dots, N$, generated by the $C^1(I)$-function λ in the sense of $x_i = \lambda(i/N)$. The function λ is defined as follows:

$$\lambda(t) = \begin{cases} \psi(t), & t \in [0, \alpha], \\ \psi(\alpha) + \psi'(\alpha)(t - \alpha), & t \in [\alpha, 1], \end{cases}$$

where

$$\psi(t) = a\varepsilon\phi(t), \quad \phi(t) = \frac{t}{q - t}, \quad t \in [0, \alpha],$$

a and q are fixed mesh parameters, $a > 0$ and $q \in (0, 1)$, and α is the t-coordinate of the point where the tangent line from the point $(1, 1)$ touches the curve $x = \psi(t)$. The equation of this tangent line is given by the second part of λ for $t \in [\alpha, 1]$. The point α satisfies the equation

$$\psi(\alpha) + \psi'(\alpha)(1 - \alpha) = 1,$$

which reduces to the quadratic equation and α is easy to find,

$$\alpha = \frac{q - \sqrt{a\varepsilon q(1 - q + a\varepsilon)}}{1 + a\varepsilon}.$$

We can see that α exists in the interval $(0, q)$ if $a\varepsilon < q$, which is what we assume from now on.

The original Bakhvalov mesh[2] has a logarithmic function $\phi_B(t) := -\ln\left(1 - \dfrac{t}{q}\right)$ for the function ϕ. The inverse of the function

ϕ_B is $\phi_B^{-1} = q\left(1 - e^{-t}\right)$, which corresponds to the the exponential-layer function. If we replace e^{-t} with its Padé approximation $\dfrac{1}{1+t}$, we obtain the function ϕ that we use here. This function ϕ was proposed by Vulanović in [19], which is why we refer to the mesh I^N as the Vulanović-Bakhvalov mesh (VB mesh).

Let $h_i = x_i - x_{i-1}$, $i = 1, 2, \ldots, N$, and $\hbar_i = (h_i + h_{i+1})/2$, $i = 1, 2, \ldots, N-1$. We also define $x_{i+1/2} = x_i + h_{i+1}/2$, $i = 0, 1, \ldots, N-1$.

We use C to denote a generic positive constant that is independent of both ε and N. Some specific constants of this kind will be subscripted. We have that $0 < \lambda'(t) \le C$, $t \in I$, which implies that $h_i \le CN^{-1}$, $i = 1, 2, \ldots, N$. It is also easy to see that $\lambda''(t) \ge 0$, $t \in I$, so that $h_i \le h_{i+1}$, $i = 1, 2, \ldots, N-1$. Moreover, when $\varepsilon \to 0$, α behaves like $q - C\sqrt{\varepsilon}$, which gives that $N^{-1} \le \min\{\alpha, 1-\alpha\}$ provided N is sufficiently large. Then it follows that

$$h_1 = \psi(N^{-1}) \le C\varepsilon N^{-1} \text{ and } h_N \ge N^{-1}\psi'(\alpha) \ge CN^{-1}. \tag{6}$$

From now on, we assume that $N \ge N_*$, where N_* is a sufficiently large constant independent of ε.

Let W^N, U^N, etc., denote mesh functions on I^N. Any mesh function is identified with an $(N+1)$-dimensional column vector, $W^N = [W_0^N, W_1^N, \ldots, W_N^N]^T$. We use the usual maximum vector norm,

$$\left\|W^N\right\| = \max_{0 \le i \le N} |W_i^N|,$$

and the the matrix norm, also denoted by $\|\cdot\|$, which is induced by this vector norm. For any function g defined on I, we write g_i instead of $g(x_i)$ and g^N for the corresponding mesh function. Thus, $u^N = [u_0, u_1, \ldots, u_N]^T$, where u is the solution of the continuous problem (1).

Let us now introduce the finite-difference operators that we are going to use to discretize the differential operator L. The scheme for u'' is the standard central scheme

$$D''W_i^N = \frac{1}{\hbar_i}\left(\frac{W_{i+1}^N - W_i^N}{h_{i+1}} - \frac{W_i^N - W_{i-1}^N}{h_i}\right).$$

For u', we use either the central scheme D' or the upwind scheme D'_+,

$$D'W_i^N = \frac{W_{i+1}^N - W_{i-1}^N}{2\hbar_i}, \quad D'_+W_i^N = \frac{W_{i+1}^N - W_i^N}{h_{i+1}}.$$

Finally, we also need D° to approximate $u_{i+1/2}$,

$$D^\circ W_i^N = \frac{W_i^N + W_{i+1}^N}{2}.$$

The central, L_c^N, and midpoint upwind, L_m^N, discretizations of the operator L are

$$L_c^N W_i^N := -\varepsilon D''W_i^N - b_i D'W_i^N + c_i W_i^N$$

and

$$L_m^N W_i^N := -\varepsilon D''W_i^N - b_{i+1/2} D'_+W_i^N + c_{i+1/2} D^\circ W_i^N.$$

Let $b(x) \le B$, $x \in I$, and let $\rho_i = \dfrac{Bh_i}{\varepsilon}$, $i = 1, 2, \ldots, N$. Then, either $\rho_i \le 1$ for all i or there exists an index J such that $1 \le J \le N-1$ and $\rho_J \le 1 < \rho_{J+1}$. Indeed, $\rho_1 \le 1$ because of the first inequality in (6). We can now define our hybrid scheme,

$$L^N W_i^N := \begin{cases} L_c^N W_i^N & \text{if } i = 1, 2, \ldots, J, \\ L_m^N W_i^N & \text{if } i = J+1, J+2, \ldots, N-1. \end{cases} \tag{7}$$

Remark 2.1. *If $\rho_N \le 1$, then $\rho_i \le 1$ for all i and the scheme in (7) should be understood as $L^N \equiv L_c^N$. Because of the second inequality in (6), this can only happen when $N^{-1} \le C_0\varepsilon$, which is not very likely in practice, but it is a theoretical possibility. Otherwise, if $J < N-1$, we have that $\varepsilon \le C_1 N^{-1}$, which is more realistic.*

3. The Numerical Solution of the Auxiliary Problem

Let us now consider the auxiliary problem (5). For simplicity, let us in this section drop the subscript γ from the solution of (5) and the components of its Kellogg-Tsan decomposition. Then, $z = u - v$ solves the following problem

$$Lz = g(x) := f(x) - \mathcal{L}v(x), \quad x \in I, \quad -z'(0) = 0, \quad z(1) = -v(1). \tag{8}$$

This is the problem we solve numerically using the hybrid scheme (7) at the points $x_1, x_2, \ldots, x_{N-1}$ of the VB mesh.

The scheme at $x_0 = 0$ is constructed as follows. Since $z'(0) = 0$, from $(Lz)(0) = g(0)$, we have $z''(0) = [c(0)z(0) - g(0)]/\varepsilon$. We use

Taylor's expansion next to get

$$D'_+ z_0 = z'_0 + \frac{h_1}{2} z''_0 + \frac{h_1^2}{6} z'''(\theta) = \frac{h_1}{2\varepsilon}(cz-g)_0 + \frac{h_1^2}{6} z'''(\theta), \tag{9}$$

where θ is a point in $(0, x_1)$. Therefore,

$$-\tilde{D}'_+ Z_0^N := \frac{1}{h_1}\left[\left(1 + \frac{h_1^2 c_0}{2\varepsilon}\right) Z_0^N - Z_1^N\right] = \frac{h_1 g_0}{2\varepsilon}$$

is a second-order discretization of the boundary condition $-z'(0) = 0$.

This is, then, how we discretize the problem (8):

$$\begin{aligned} -\tilde{D}'_+ Z_0^N &= \frac{h_1 g_0}{2\varepsilon} \\ L_c^N Z_i^N &= g_i, \quad i = 1, 2, \dots, J, \\ L_m^N Z_i^N &= g_{i+1/2}, \quad i = J+1, J+2, \dots, N-1, \\ Z_N^N &= -v(1). \end{aligned} \tag{10}$$

According to Remark 2.1, if $J = N-1$, the scheme L_m^N is not used at all in the above discretization.

Let $A_N = [a_{ij}]$ be the matrix of the system (10). It is easy to see that A_N is an L-matrix, that is, $a_{ii} > 0$ and $a_{ij} \le 0$ if $i \ne j$, for all $i, j = 0, 1, \dots, N$, provided $N \ge N_*$. Moreover, A^N is also nonsingular and all elements of $(A^N)^{-1}$ are non-negative. This follows from the fact that

$$(A^N y^N)_i \ge \min\{1, \beta\}, \quad \text{where} \quad y_i^N = 2 - x_i, \quad i = 0, 1, \dots, N.$$

The final conclusion is that the discrete system (10) is stable uniformly in ε,

$$\|A_N^{-1}\| \le C,$$

and that it has a unique solution Z^N.

The discretization (10) is also second-order consistent uniformly in ε. This follows from the standard proof technique which is based on the properties of the VB mesh and the derivative estimates (4) for the function z, see [19, 20] for instance. It is essential that z is a little less stiff than u when $\varepsilon \to 0$; the proof cannot work for u. When the scheme L_c^N is used, the proof of second-order consistency uniform in ε is straightforward. As for L_m^N, it should be mentioned that $D''u_i$ is only a first-order approximation of $u''_{i+1/2}$. However, when the midpoint upwind discretization is used, it follows that $\varepsilon \le Ch_i$. This increases the accuracy of $\varepsilon D''u_i$ to the second order. The only new detail regarding the discretization (10) is the consistency error at $x_0 = 0$. Because of (9), (4), and the first inequality in (6), we have

$$|z'_0 - D'_+ z_0| \le C h_1^2 \varepsilon^{-2} \le CN^{-2}.$$

With this, we correct the scheme used in [20], where a first-order discretization at $x_0 = 0$ is mistaken for a second-order one. Therefore, we have the following second-order ε-uniform convergence result.

Theorem 3.1. *Assume that $b, c, f \in C^3(I)$ and that b and c satisfy* (2). *Let z be the solution of the continuous problem* (8) *and let Z^N be the solution of the discrete problem* (10) *on the VB mesh with $N \ge N_*$. Then,*

$$\|z^N - Z^N\| \le CN^{-2}.$$

Equivalently, if u is the solution of the continuous problem (5) *and $U^N = v^N + Z^N$, then*

$$\|u^N - U^N\| \le CN^{-2}.$$

Remark 3.2. *The problem* (5) *is a singular perturbation problem that may be of interest in its own right. For instance, its numerical solution is discussed in [23, 5]. In the latter paper, the upwind scheme on the Shishkin mesh is used and it is proved that the error of the numerical solution can be estimated by $CN^{-1}\ln N$. We are here interested in the problem* (5) *mainly as an auxiliary problem for solving* (1).

4. The Numerical Solution of the Original Problem

We first state a lemma about the solution of the homogeneous problem of type (5).

Lemma 4.1. *Let the conditions in* (2) *be satisfied and let u_γ be a $C^2(I)$-solution of the problem* (5) *with $\gamma > 0$ and $f \equiv 0$. Then $u_\gamma(0) \ge m$, where m is a positive constant independent of ε.*

Proof. For the technique used in the proof, see [20]. Consider the function

$$\omega(x) = \frac{\gamma}{R}\left(e^{-Rx/\varepsilon} - e^{-R/\varepsilon}\right), \quad R = \frac{1}{2}\left(B + \sqrt{B^2 + 4G\varepsilon}\right),$$

where $c \le G$ on I (recall that $b \le B$). We use the inverse monotonicity of the differential operator L, accompanied with the boundary conditions in (5), to show that $u_\gamma \ge \omega$ on I. The assertion then follows from $\omega(0) \ge m$.

First, we have that $\omega'(x) = -\gamma\varepsilon^{-1}e^{-Rx/\varepsilon}$, thus $-\varepsilon\omega'(0) = \gamma = -\varepsilon u'(0)$. Second, $\omega(1) = 0 = u(1)$. Finally,

$$L\omega = \frac{\gamma}{\varepsilon R}[-R^2 + b(x)R + \varepsilon c(x)]e^{-Rx/\varepsilon} - \frac{\gamma}{R}c(x)e^{-R/\varepsilon} \le 0 = Lu_\gamma, \quad x \in I,$$

because $-R^2 + b(x)R + \varepsilon c(x) \le -R^2 + BR + \varepsilon G = 0$. □

We next consider two problems of type (5) with two different values of γ, γ_1 and γ_2. Let their $C^2(I)$-solutions be u^1 and u^2, where $u^j = u_{\gamma_j}$, $j = 1, 2$. Without loss of generality, we can assume that $\gamma_2 > \gamma_1$. Then, if we apply Lemma 4.1 to $u^2 - u^1$, we get the following result.

Lemma 4.2. *Let the conditions in* (2) *be satisfied and let u^1 and u^2 be the $C^2(I)$-solutions of the problem* (5) *with different respective values of $\gamma = \gamma_1$ and $\gamma = \gamma_2$. Then there exists a positive constant δ independent of ε such that $|u^2(0) - u^1(0)| \ge \delta$.*

Under the conditions of Lemma 4.2, it is possible to form the following linear combination of u^1 and u^2:

$$\ell(u^1, u^2)(x) := \frac{u^2(0)u^1(x) - u^1(0)u^2(x)}{u^2(0) - u^1(0)}. \tag{11}$$

Lemma 4.3. *Let the conditions of Lemma 4.2 be satisfied. Then the $C^2(I)$-solution of the problem* (1) *is $u = \ell(u^1, u^2)$.*

Proof. It is easy to see that $\ell(u^1, u^2)(x) = 0$ for $x = 0, 1$. Moreover, $L\ell(u^1, u^2) = f(x)$. Since the problem (1) has a unique solution u, this solution and $\ell(u^1, u^2)$ must be identical. □

We obtain numerical approximations of u^1 and u^2 using the method of Section 3. Let $Z^{1,N}$ and $Z^{2,N}$ be the corresponding solutions of the discrete problem (10) and let

$$U^{j,N} = v^N_{\gamma_j} + Z^{j,N}, \quad j = 1, 2. \tag{12}$$

Theorem 3.1 and Lemma 4.2 imply that for sufficiently large N_* independent of ε there exists a positive constant η such that $|U_0^{2,N} - U_0^{1,N}| \ge \eta$, where η is independent of ε and N. We can then form a linear combination of the two numerical solutions, which is analogous to (11),

$$U^N := \frac{U_0^{2,N}U^{1,N} - U_0^{1,N}U^{2,N}}{U_0^{2,N} - U_0^{1,N}}. \tag{13}$$

The following theorem is the main result of the paper.

Theorem 4.4. *Assume that $b, c, f \in C^3(I)$ and that b and c satisfy* (2)*. Let u be the solution of the continuous problem* (1) *and let $Z^{1,N}$ and $Z^{2,N}$ be the solutions of the discrete problem* (10) *on the VB mesh with $N \ge N_*$ and with different respective values of $\gamma = \gamma_1$ and $\gamma = \gamma_2$. Then the mesh function U^N defined by* (13) *and* (12) *satisfies*

$$\|u^N - U^N\| \le CN^{-2}.$$

Proof. The proof of this theorem is not given in [20]. Here, we provide some details.

We express u^N using the right-hand side of (11) and U^N as in (13) to get

$$\|u^N - U^N\| \le P + Q,$$

where

$$P = \left\| \frac{u_0^2 u^1}{u_0^2 - u_0^1} - \frac{U_0^2 U^1}{U_0^2 - U_0^1} \right\|, \quad Q = \left\| \frac{u_0^1 u^2}{u_0^2 - u_0^1} - \frac{U_0^1 U^2}{U_0^2 - U_0^1} \right\|,$$

and we omitted the superscript N everywhere for simplicity. We prove below that $\|P\| \le CN^{-2}$.

We have

$$\begin{aligned}
\|P\| &\le \frac{1}{\delta\eta}\|(U_0^2 - U_0^1)u_0^2 u^1 - (u_0^2 - u_0^1)U_0^2 U^1\| \\
&\le C\|U_0^2 u_0^2(u^1 - U^1) + U_0^2 u_0^1 U^1 - U_0^1 u_0^2 u^1\| \\
&\le CN^{-2} + C\|U_0^2 u_0^1 U^1 - U_0^1 u_0^2 u^1\|.
\end{aligned}$$

Table 1. Errors E^N and rates R^N for the method of Section 4 applied to the test problem (14).

ε	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
1e-02	8.28e-04	2.63e-04	1.08e-04	3.75e-05	1.58e-05	3.96e-06	9.90e-07
	1.65	1.29	1.53	1.24	2.00	2.00	
1e-04	7.13e-04	1.79e-04	4.54e-05	1.17e-05	3.08e-06	8.43e-07	2.47e-07
	1.99	1.98	1.95	1.93	1.87	1.77	
1e-06	7.59e-04	1.90e-04	4.74e-05	1.19e-05	2.96e-06	7.42e-07	1.86e-07
	2.00	2.00	2.00	2.00	2.00	2.00	
1e-08	7.64e-04	1.91e-04	4.77e-05	1.19e-05	2.98e-06	7.45e-07	1.86e-07
	2.00	2.00	2.00	2.00	2.00	2.00	
1e-10	7.65e-04	1.91e-04	4.77e-05	1.19e-05	2.98e-06	7.46e-07	1.86e-07
	2.00	2.00	2.00	2.00	2.00	2.00	

The desired estimate then follows from

$$\begin{aligned}\|U_0^2u_0^1U^1 - U_0^1u_0^2u^1\| &\le \|U_0^2u_0^1U^1 - u_0^2u_0^1U^1\| + \|u_0^1u_0^2U^1 - U_0^1u_0^2U^1\| \\ &\quad + \|U_0^1u_0^2U^1 - U_0^1u_0^2u^1\| \\ &\le C(|U_0^2 - u_0^2| + |u_0^1 - U_0^1| + \|U^1 - u^1\|) \le CN^{-2}.\end{aligned}$$

The proof of $\|Q\| \le CN^{-2}$ is analogous. □

5. Numerical Results

To illustrate the result of Theorem 4.4, we consider two test problems. The first one is used in [21],

$$-\varepsilon u'' - (x+1)u' + u = f(x), \quad x \in I, \quad u(0) = u(1) = 0, \tag{14}$$

where the function f is chosen so that

$$u(x) = e^{-x/\varepsilon} - e^x + (e - e^{-1/\varepsilon})x.$$

We apply the method described in Section 4, solving the discrete problem (10) twice, with $\gamma = 0$ and $\gamma = 1$. Note that the two different values of γ only change the right-hand side of the discrete system. Its matrix remains the same, so the two discrete problems can even be solved simultaneously. When forming the quantity ρ_i, which determines when the switch from the central scheme to the midpoint upwind one takes place, we used $B = \max_{x\in I} b(x) = 2$.

We calculate the error $E^N = \|U^N - u^N\|$ of the numerical solution U^N and we estimate the rate of convergence by $R^N = \log_2 E^N - \log_2 E^{2N}$. The VB mesh is used with $a = 2$ and $q = 0.5$. The results are presented in Table 1. The uniformity of the errors when $\varepsilon \to 0$ can be observed. The rates of convergence are close or exactly equal to the theoretical value of 2 when $\varepsilon \le 10^{-4}$. When $\varepsilon = 10^{-2}$, the expected values of R^N can be observed for $N \ge 256$ (it is interesting to mention that $\varepsilon = 10^{-2}$ is the only case in Table 1 when the hybrid scheme reduces to the central scheme, cf. Remark 2.1). In general, the rate of convergence is an asymptotic property as $N \to \infty$, so it is possible that it only shows for sufficiently large N. This can also be seen in the other two tables in this section.

The hybrid scheme (7) can be applied to the continuous problem (1) directly. Second-order convergence uniform in ε can also be proved in that case but the method of proof is more complicated; see [18] and [14] for different proofs when the discretization mesh is the Shishkin mesh. It is interesting to compare the method of Section 4 to the direct discretization of (1) using the hybrid scheme (7) on the VB mesh. The results for the latter are given in Table 2. Theoretically, the rates of convergence of the two methods are the same. The rates in both Tables 1 and 2 are close to 2 when N increases. However, the errors are considerably smaller for our method, undoubtedly because the boundary-layer component of the Kellogg-Tsan decomposition does not contribute to the error, which is not the case with the other method.

Our second example is taken from [7],

$$-\varepsilon u'' - (1-x)u' + 2u = 3(1-x), \quad x \in I, \quad u(0) = 2, \quad u(1) = 0. \tag{15}$$

The exact solution is $u(x) = (1-x)\left[1 + e^{-x(2-x)/(2\varepsilon)}\right]$. This problem is more general than (1)–(2) since $b(x) \ge b(1) = 0$, $x \in I$. Nevertheless, b is positive to the left of $x = 1$ and the solution behaves like in (3)–(4). The same problem is also used in [10] to illustrate the theoretical results obtained there for a problem of the (1)–(2) type. To apply our method to problem (15), we transform it using the substitution $\bar{u} = u + 2(x-1)$, which changes the boundary conditions to the homogeneous ones. In general, when the original Dirichlet boundary conditions are not homogeneous, an analogous substitution (a linear function of x added to u) should be made and then the method can be applied to the transformed problem with homogeneous boundary conditions. The substitution only changes the right-hand side of the differential equation.

Table 2. Errors E^N and rates R^N for the hybrid scheme (7) on the VB mesh applied directly to the test problem (14).

ε	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
1e-02	5.40e-02	1.24e-02	7.63e-04	1.24e-04	5.15e-05	1.29e-05	3.22e-06
	2.12	4.02	2.63	1.26	2.00	2.00	
1e-04	6.38e-02	1.42e-02	8.38e-04	1.24e-04	3.60e-05	8.99e-06	2.22e-06
	2.17	4.08	2.75	1.79	2.00	2.02	
1e-06	6.43e-02	1.42e-02	8.41e-04	1.24e-04	3.61e-05	9.09e-06	2.27e-06
	2.17	4.08	2.76	1.78	1.99	2.00	
1e-08	6.43e-02	1.42e-02	8.41e-04	1.24e-04	3.61e-05	9.08e-06	2.27e-06
	2.17	4.08	2.76	1.78	1.99	2.00	
1e-10	6.43e-02	1.42e-02	8.41e-04	1.24e-04	3.61e-05	9.08e-06	2.27e-06
	2.17	4.08	2.76	1.78	1.99	2.00	

Table 3. Errors E^N and rates R^N for the method of Section 4 applied to the test problem (15).

ε	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
1e-02	3.86e-04	2.03e-04	2.32e-05	5.36e-06	1.33e-06	3.34e-07	8.36e-08
	0.93	3.13	2.11	2.00	2.00	2.00	
1e-04	5.02e-06	2.41e-06	4.09e-07	5.79e-08	1.40e-08	3.50e-09	8.75e-10
	1.06	2.56	2.82	2.04	2.00	2.00	
1e-06	5.07e-08	2.41e-08	4.09e-09	5.79e-10	1.40e-10	3.50e-11	8.80e-12
	1.07	2.56	2.82	2.05	2.00	2.00	
1e-08	5.07e-10	2.41e-10	4.09e-11	5.79e-12	1.39e-12	3.57e-13	1.43e-13
	1.07	2.56	2.82	2.05	1.96	1.32	

We used the same VB mesh as for the problem (14) and the hybrid scheme with $B=1$. Our method works extremely well for (15). Table 3 shows that the errors decrease together with ε, indicating that they behave like $C\varepsilon N^{-2}$. This is so because the solution of the corresponding reduced problem,

$$-(1-x)u_r' + 2u_r = 3(1-x), \quad x \in I, \quad u(1) = 0,$$

is simply $u_r = 1-x$ and our scheme is exact for any linear function. Thus, the only component of the solution u that produces any error in our numerical method is the exponential boundary-layer function within z. If we denote this function by $\hat{z}$, based on (4), we can expect that

$$|\hat{z}^{(j)}(x)| \le C\varepsilon^{1-j}e^{-\beta x/\varepsilon}, \quad x \in I, \quad j = 0, 1, \ldots, k+1.$$

This explains the presence of the ε-factor in the error. As the accuracy of the method approaches machine precision for the smallest values of ε and the largest values of N, the errors become unreliable. This is why the results for $\varepsilon = 10^{-10}$ are omitted from Table 3. Already for $\varepsilon = 10^{-8}$ and $N = 1024$, the error is affected by machine precision and the corresponding value of R^N is only 1.32.

When the test problem (15) is discretized directly by the hybrid scheme (7) on the VB mesh, the results are similar to those in Table 2 and cannot compete at all with those of Table 3. The same can be said about the results reported in [7, 10] for the respective numerical methods considered there. Of course, such excellent results cannot be expected of our method in general. However, this numerical experiment motivates the idea to include the reduced solution in the decomposition of the solution u, together with the components of the Kellogg-Tsan decomposition, so that the only error of the numerical method is that of approximating $\hat{z}$. We are currently working on this topic.

6. Conclusion

Our work is related to the question of how numerical methods for solving singularly perturbed convection-diffusion problems of type (1) can benefit from the Kellogg-Tsan decomposition of the continuous solution. We followed the idea, introduced in [20], that renders the boundary-layer component of the decomposition completely known. This leaves the smooth component to be solved for. The theoretical benefit of this is that the proof of ε-uniform convergence is simplified. If the discretization scheme used is stable uniformly in ε, the only thing left to prove is ε-uniform consistency. Otherwise, without the Kellogg-Tsan decomposition, convection-diffusion problems require special proof methods to get ε-uniform convergence in the absence of ε-uniform consistency.

The boundary-layer component is also needed for the construction of fitted discretization schemes, but the way we use it allows for standard schemes on layer-adapted meshed. In this paper, we considered a second-order hybrid scheme. However, our approach can be extended to schemes of order higher than two, like the one in [21], for which fitted schemes are complicated and hard to analyze.

The practical benefit of the use of the Kellogg-Tsan decomposition is that the boundary-layer component does not contribute to the error of the numerical solution. Although this does not change the order of ε-uniform convergence, we can expect the errors to be less than the errors obtained when the same discretization is applied to the continuous problem without decomposing its solution.

The numerical results presented in [20] and here confirm this expectation. However, experimenting with different test problems and values of ε and the mesh parameters N, a and q, we have occasionally come across the situation when this improvement is not present. Therefore, any proof, if at all possible, that would guarantee this benefit of the Kellogg-Tsan decomposition, would have to assume certain conditions on the problem (1)–(2) and its discretization.

Despite its attractive aspects, the method of [20] is not well-known. This is why we undertook this reiteration, in which we also revised the original paper improving and correcting some of its parts. As for the continuation of this work, one of our test problems indicated that the method can be considerably improved by involving the reduced solution in the decomposition of the Kellogg-Tsan type. The results of our research in this direction will be reported elsewhere.

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Conflicts of Interest

The authors declare no conflict of interest.

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