A preconditioning-based analysis for a Bakhvalov-type mesh

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Abstract

A new preconditioning-based parameter-uniform convergence analysis is presented for one-dimensional singularly perturbed convection-diffusion problems discretized by an upwind difference scheme on a Bakhvalov-type mesh. The proof utilizes the classical convergence principle: uniform stability and uniform consistency imply uniform convergence, which can only be used after applying an appropriate preconditioner to the discrete operator.

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1 Introduction

The purpose of this article is to provide a new theoretical analysis for the one-dimentional singularly perturbed convection-diffusion problem

$$\mathcal{L}u := -\varepsilon u'' - b(x)u' + c(x)u = f(x)\,, \quad x \in (0,1)\,, \quad u(0) = u(1) = 0\,, \quad (1)$$

where ε is a positive perturbation parameter, $0 < \varepsilon \leqslant 1$. We assume that the functions b, c and f are sufficiently smooth, and that

$$b(x) \geqslant \beta > 0\,, \quad c(x) \geqslant 0 \quad \text{for } x \in I := [0,1]\,.$$

When ε is small, the problem (1) is convection-dominated and an exponential boundary layer typically arises when the flow (presented by the convection term $b(x) > 0$) travels towards the boundary. Then, the boundary value problem (1) has a unique solution $u \in C^2(I)$.

Because of the presence of such exponential layers, special numerical methods are derived to meaningfully resolve the layers and to achieve *parameter-robust* convergence (or ε-uniform convergence). The use of layer-adapted meshes, in conjuntion with either finite-difference or finite-element discretizations, is one of the most frequently used approaches to achieve the goal. In 1969, Bakhvalov [1] introduced the first layer-fitted mesh by applying the inverse of the exponential-layer function into its mesh-generating function. About two decades later, Shishkin [12] proposed the piecewise-uniform mesh which received much interest due to its simplicity in construction and analysis. Nevertheless, as a trade-off, the convergence rate of the Shishkin mesh is usually sub-optimal compared to that of the Bakhvalov mesh. For further important concepts and advances in the field of numerical analysis of singular perturbation problems, we refer the reader to the works by Linß [3], Roos et al. [11], Roos and Stynes [10], and Stynes [13].

Even when layer-adapted meshes are used, error analyses of finite-difference methods for problem (1) are still challenging. This is because the derivatives of the solution behave like $\mathcal{O}(\varepsilon^{-k})$, where k is some positive integer. Furthermore, the truncation errors of an upwind difference scheme discretized on the Shishkin mesh are not ε-uniformly consistent, and of order $\mathcal{O}(\varepsilon^{-1}N^{-1}\ln N)$ where N is the discretization parameter [6, 14, for numerical observations of this phenomenon]. Because of these issues, special techniques are devised to prove ε-uniform convergence for finite-difference methods on layer-adapted meshes. These include the hybrid-stability approach [4], truncation-error and barrier functions [7, 8], and the grid transformation [5, Chapter 7]. Recently, another method developed for proving uniform convergence on the Shishkin mesh is the preconditioning technique to enable the classical principle: "ε-uniform stability and ε-uniform consistency imply ε-uniform convergence". This idea was first introduced by Vulanović and Nhan [14] and extended further by Nhan et al. [6] and Vulanović and Nhan [15] to handle hybrid higher-order finite-difference schemes, but only on the piecewise-uniform Shishkin mesh.

The goal of this article is to show that it is possible to generalize the preconditioning-driven analysis to an exponentially graded Bakhvalov-type mesh. In particular, our motivation comes from the fact that this new approach has proven its salient advantage over the aforementioned proofs. As shown by Vulanović and Nhan [15], an almost-third-order difference scheme can be only analysed by the preconditioning. We emphasise that our intended contribution is not the main uniform convergence theorem (already proven by other methods), but rather it is the novel analysis which makes use of the preconditioning approach for a Bakhvalov-type mesh. Our result might be employed to analyse more complicated higher-order schemes, similar to Vulanović and Nhan [15] but on Bakhvalov-type meshes.

In the next section, we describe an upwind discretization on a Bakhvalov-type mesh. Section 3 introduces an appropriate preconditioner to scale the discretized system and obtain the ε-uniform stability. Finally, the preconditioned consistency error is analysed and the uniform convergence result is derived.

2 Upwind scheme on a Bakhvalov-type mesh

The following decomposition of u is often used in the error analysis of numerical methods for problem (1) [3, Theorem 3.48]:

$$u(x) = s(x) + y(x)\,, \tag{2}$$

$$|s^{(k)}(x)| \leqslant C\left(1+\varepsilon^{2-k}\right), \quad |y^{(k)}(x)| \leqslant C\varepsilon^{-k}e^{-\beta x/\varepsilon}, \quad x \in I\,, \quad k = 0,1,2,3, \tag{3}$$

where C is a positive generic constant independent of ε and N. Moreover, the layer component y satisfies a homogeneous differential equation:

$$\mathcal{L}y(x) = 0\,, \quad x \in (0,1)\,. \tag{4}$$

Let I^N denote an arbitrary mesh with mesh points x_i, $i = 0,1,\dots,N$, such that $0 = x_0 < x_1 < \cdots < x_N = 1$. Let $h_i = x_i - x_{i-1}$, $i = 1,2,\dots,N$, be the mesh-step sizes and let $\bar{h}_i = (h_i + h_{i+1})/2$. Mesh functions on I^N are denoted by, for example, $W^N = \left(W_i^N\right)$, $U^N = \left(U_i^N\right)$. If g is a function defined on I, then $g_i := g(x_i)$, and $g_i^N := g^N(x_i)$ for the corresponding mesh function. We use the maximum norm of W^N, $\left\|W^N\right\| = \max_{0\leqslant i\leqslant N} |W_i^N|$. The matrix norm induced by the maximum vector norm is also denoted by $\|\cdot\|$.

The upwind finite-difference scheme is used to discretize the problem (1) on I^N:

$$U_0^N = 0\,, \quad U_N^N = 0\,,$$
$$\mathcal{L}^N U_i^N := -\varepsilon D'' U_i^N - b_i D^+ U_i^N + c_i U_i^N = f_i\,, \quad i = 1,2,\dots,N-1, \tag{5}$$

where

$$D'' U_i^N = \frac{1}{\bar{h}_i}\left(D^+U_i^N - D^-U_i^N\right),$$
$$D^+U_i^N = \frac{U_{i+1}^N - U_i^N}{h_{i+1}}\,, \quad D^-U_i^N = \frac{U_i^N - U_{i-1}^N}{h_i}\,.$$

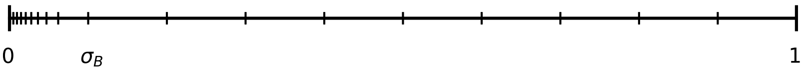

Figure 1: The Bakhvalov-type mesh defined by (6) and (7) for one-dimensional convection-diffusion problems (1).

We are interested in the Bakhvalov-type mesh introduced by Boglaev [2]:

$$x_i = \begin{cases} \varepsilon\phi_B(t_i)\,, & i = 0, 1, \ldots, J, \\ \sigma_B + 2(1-\sigma_B)(t_i - 1/2)\,, & i = J+1, J+2, \ldots, N, \end{cases} \tag{6}$$

with $\phi_B(t) := -\ln[1 - 2(1-\varepsilon)t]$, $t_i = i/N$, $J = N/2$, and transition point defined by

$$\sigma_B := \varepsilon \ln(1/\varepsilon) = x_J\,. \tag{7}$$

The mesh, plotted in Figure 1, is gradually graded in the layer region with $h_{i-1} \leqslant h_i$, $i = 2, \ldots, J$, and is equally spaced with the step size $H := h_i$, $i = J+1, \ldots, N$. This Bakhvalov-type mesh shares the following characteristics with the original Bakhvalov mesh: a logarithmic function to generate the points in the layer, and a transition point (7) of the order $\mathcal{O}(-\varepsilon \ln \varepsilon)$. Furthermore, the last mesh step, $h_J := \varepsilon \ln(1 + 2(1-\varepsilon)/(\varepsilon N))$, in the layer region tends to zero as $\varepsilon \to 0$; but more slowly than that of the Shishkin-type meshes (in the sense described by Roos and Linß [9]) due to the logarithmic factor. This is why the analysis for the Bakhvalov-type meshes is usually more delicate than the Shishkin-type meshes [8].

Lemma 1. *The mesh widths in the layer regions of the Bakhvalov-type mesh defined by* (6) *and* (7) *satisfy*

$$h_{i-1} \leqslant h_i \leqslant CN^{-1}\,, \quad i = 2, 3, \ldots, J, \tag{8}$$

and in particular

$$h_i \leqslant \varepsilon\,, \quad i = 1, 2, \ldots, J-1. \tag{9}$$

Furthermore,

$$e^{-\beta x_{J-1}/\varepsilon} = \left(\varepsilon + 2(1-\varepsilon)N^{-1}\right)^{\beta} \leqslant \left(\varepsilon + 2N^{-1}\right)^{\beta} \tag{10}$$

and

$$e^{-\beta x_J/\varepsilon} = \varepsilon^{\beta}\,. \tag{11}$$

Proof: By the definition of $\phi_B(t)$, we have $\phi_B'(t) = 2(1-\varepsilon)/[1-2(1-\varepsilon)t]$, thus $\phi_B(t)$ is monotonically increasing for $t \in [0, 1/2]$. Therefore, $h_{i-1} \leqslant h_i$ $i = 2, 3, \ldots, J-1$, and

$$h_J = \varepsilon \int_{t_{J-1}}^{t_J} \phi_B'(s)\,ds \leqslant \frac{\varepsilon}{N}\phi_B'(t_J) = \frac{\varepsilon}{N}\cdot\frac{2(1-\varepsilon)}{1-2(1-\varepsilon)t_J} \leqslant \frac{\varepsilon}{N}\cdot\frac{2}{\varepsilon} = 2N^{-1},$$

which gives (8).

For the bound in (9), we have

$$\begin{aligned} h_i = \varepsilon \int_{t_{i-1}}^{t_i} \phi_B'(s)\,ds &\leqslant \frac{\varepsilon}{N} \max_{t\in[t_{i-1},t_i]} \phi_B'(t) = \frac{\varepsilon}{N}\cdot\frac{2(1-\varepsilon)}{1-2(1-\varepsilon)t_i} \\ &\leqslant \frac{\varepsilon}{N}\cdot\frac{2}{1/(1-\varepsilon)-2t_{J-1}} \leqslant \frac{\varepsilon}{N}\cdot N = \varepsilon\,. \end{aligned}$$

We use the definitions of x_{J-1} in (6) and x_J in (7) to get the inequality (10) and the equality (11). ♠

When the upwind scheme (5) is discretized on the Bakhvalov-type mesh, the linear system in matrix form is

$$A_N U^N = f^N,$$

where $A_N = [a_{ij}]$ is a tridiagonal matrix with $a_{00} = 1$ and $a_{NN} = 1$ being the only nonzero elements in the 0th and Nth rows, respectively, and where

$f^N = [0, f_1, f_2, \dots, f_{N-1}, 0]^T$. Let the entries of A_N be denoted by $a_{i,j}$, the nonzero ones being

$$l_i := a_{i-1,i} = \begin{cases} -\frac{\varepsilon}{\bar{h}_i h_i}, & 1 \leqslant i \leqslant J, \\ -\frac{\varepsilon}{H^2}, & J+1 \leqslant i \leqslant N-1, \end{cases}$$

$$r_i := a_{i,i+1} = \begin{cases} -\frac{\varepsilon}{\bar{h}_i h_{i+1}} - \frac{b_i}{h_{i+1}}, & 1 \leqslant i \leqslant J-1, \\ -\frac{\varepsilon}{\bar{h}_i H} - \frac{b_i}{H}, & i = J, \\ -\frac{\varepsilon}{H^2} - \frac{b_i}{H}, & J+1 \leqslant i \leqslant N-1, \end{cases}$$

and

$$d_i := a_{ii} = \begin{cases} 1, & i = 0, \\ -l_i - r_i + c_i, & 1 \leqslant i \leqslant N-1, \\ 1, & i = N. \end{cases}$$

3 Preconditioning and ε-uniform stability

The goal of this section is to precondition the discrete systems (5) in such a way that the ε-uniform stability is retained; and simultaneously, the modified consistency errors can be proven to be convergent uniformly in ε.

Let $M = \operatorname{diag}(m_0, m_1, \dots, m_N)$ be a diagonal matrix with the entries

$$m_0 = 1, \quad m_i = \frac{\bar{h}_i}{H}, \quad i = 1, \dots, J, \quad \text{and} \quad m_i = 1, \quad i = J+1, \dots, N. \tag{12}$$

Then, the left-preconditioned system is

$$\tilde{A}_N U^N = \tilde{f}^N, \tag{13}$$

where $\tilde{A}_N = MA_N$ and $\tilde{f}^N = Mf^N$. The entries of $\tilde{A}_N$ are denoted by $\tilde{a}_{ij}$, and the nonzero ones are

$$\tilde{l}_i := \tilde{a}_{i,i-1} = \begin{cases} -\frac{\varepsilon}{h_i H}, & 1 \leqslant i \leqslant J, \\ -\frac{\varepsilon}{H^2}, & J+1 \leqslant i \leqslant N-1, \end{cases}$$

$$\tilde{r}_i := \tilde{a}_{i,i+1} = \begin{cases} -\frac{\varepsilon}{h_{i+1}H} - \frac{b_i\bar{h}_i}{h_{i+1}H}\,, & 1 \leqslant i \leqslant J-1\,,\\ -\frac{\varepsilon}{H^2} - \frac{b_i\bar{h}_i}{H^2}\,, & i = J\,,\\ -\frac{\varepsilon}{H^2} - \frac{b_i}{H}\,, & J+1 \leqslant i \leqslant N-1\,, \end{cases}$$

and

$$\tilde{d}_i := \tilde{a}_{ii} = \begin{cases} 1, & i = 0\,,\\ -\tilde{l}_i - \tilde{r}_i + c_i\frac{\bar{h}_i}{H}\,, & 1 \leqslant i \leqslant J\,,\\ -\tilde{l}_i - \tilde{r}_i + c_i\,, & J+1 \leqslant i \leqslant N-1\,,\\ 1, & i = N\,. \end{cases}$$

In order to show the ε-uniform stability of the preconditioned discretization (13), we need to show that $\tilde{A}_N$ is an M-matrix. We first derive a technical lemma that results from the special structure of the Bakhvalov-type mesh.

Lemma 2. *Let $\beta > 2$ and $\Delta_i := \phi_B(t_i) - \phi_B(t_{i-1})$. Then, there exist a sufficiently large N_0, and a positive constant δ independent of both ε and N, such that for all $N \geqslant N_0$, we have*

$$S_i := \frac{\beta}{2}\left(1 + \frac{\Delta_i}{\Delta_{i+1}}\right) - \left(\frac{1}{\Delta_i} - \frac{1}{\Delta_{i+1}}\right) \geqslant \delta > 0\,, \quad i = 1,\ldots,J-1. \qquad (14)$$

Proof: From the assumption $\beta > 2$ there exists a constant $\delta > 0$, independent of ε and N, such that $\beta/2 \geqslant 1 + \delta$. Then S_i, for $1 \leqslant i \leqslant J-1$, can be bounded from below as

$$\begin{aligned} S_i &\geqslant (1+\delta)\left(1 + \frac{\Delta_i}{\Delta_{i+1}}\right) - \left(\frac{1}{\Delta_i} - \frac{1}{\Delta_{i+1}}\right)\\ &\geqslant \left(1 + \frac{\Delta_i}{\Delta_{i+1}} + \frac{1}{\Delta_{i+1}} - \frac{1}{\Delta_i}\right) + \delta = \frac{\Delta_i\Delta_{i+1} + \Delta_i^2 + \Delta_i - \Delta_{i+1}}{\Delta_i\Delta_{i+1}} + \delta\,. \end{aligned}$$

The assertion in (14) is proven if we can show that the numerator $\Delta_i\Delta_{i+1} + \Delta_i^2 + \Delta_i - \Delta_{i+1} \geqslant 0$. By direct computations, we get

$$\Delta_i = \ln\left(1 + \frac{1}{N/(2(1-\varepsilon)) - i}\right) = \ln(1+z)\,,$$

where
$$z := [N/(2(1-\varepsilon)) - i]^{-1}\,. \tag{15}$$
On the other hand, in a different form
$$\begin{aligned}\Delta_{i+1} &= \ln\left(\frac{1-2(1-\varepsilon)t_i}{1-2(1-\varepsilon)t_{i+1}}\right)\\ &= \ln\left(1-\frac{1}{N/(2(1-\varepsilon))-i}\right)^{-1} = \ln\left(\frac{1}{1-z}\right).\end{aligned}$$
Then,
$$\Delta_i\Delta_{i+1}+\Delta_i^2+\Delta_i-\Delta_{i+1} = \ln(1+z)\left(\ln\frac{1+z}{1-z}+1\right)+\ln(1-z) =: g(z).$$
The function $g(z)$ is defined for $z\in(-1,1)$. Its two roots are $z_* := 0$ and an irrational one denoted by z^*. We investigate $g(z)$ numerically on the interval $[0, 0.98]$ using Maple—a computer algebra system—and its graph on this interval is plotted in Figure 2 which shows $g(z)\geqslant 0$ for all values of z in between its two roots, z_* and z^*. Additionally, the second root z^* is numerically approximated as $z^*\approx 0.971$.

We now show that for each ε there exists a sufficiently large N such that $g(z)>0$. That is, by the definition (15) of z, the lower and upper bounds of z are
$$\begin{aligned}0<\frac{1}{N/(2(1-\varepsilon))}\leqslant z &\leqslant \frac{1}{N/(2(1-\varepsilon))-(J-1)}\\ &= \frac{1}{N/(2(1-\varepsilon))-(N/2-1)} = \frac{1}{1+\frac{N}{2}\left(\frac{1}{1-\varepsilon}-1\right)}\\ &\leqslant \frac{1}{1+N_0\eta} < z^*,\end{aligned}$$
with $\eta := \frac{1}{2}\left(\frac{1}{1-\varepsilon}-1\right)$. We can choose a sufficiently large integer N_0 to guarantee that $\frac{1}{1+N_0\eta}\leqslant 0.971 < z^*$ because $\eta>0$ for all $\varepsilon<1$. ♠

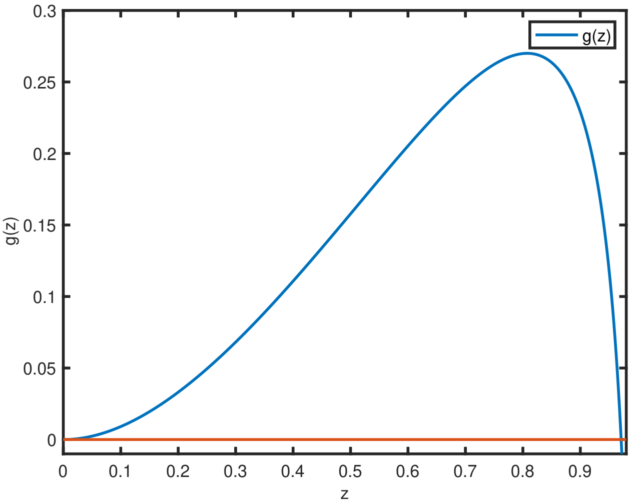

Figure 2: Graph of the function $g(z)$ for $z \in [0, 0.98]$.

Remark 3. In Lemma 2, the technical assumption $\beta > 2$ can be relaxed to $\beta > 0$ by having a user-chosen positive parameter a in the definition of the graded mesh points x_i in (6); that is, set $x_i = a\varepsilon\phi_B(t_i)$ for $i = 0, 1, \ldots, J$.

Lemma 4. *Under the assumptions of Lemma 2 , the matrix* $\tilde{A}_N$ *of the system (13) satisfies* $\|\tilde{A}_N^{-1}\| \leqslant C$.

Proof: We only outline the essential calculation here. The detailed proof can be found elsewhere [6, 14, 15]. We construct a vector $\nu = [\nu_0, \nu_1, \ldots, \nu_N]^T$ with elements

$$\nu_i = \alpha - Hi + \lambda \min\{(1+\rho)^{J-i}, 1\}, \quad i = 0, 1, \ldots, N,$$

where α and λ are appropriately chosen positive constants and $\rho = \beta H/\varepsilon$. The following argument is to prove the condition: $\tilde{l}_i\nu_{i-1} + \tilde{d}_i\nu_i + \tilde{r}_i\nu_{i+1} \geqslant \delta_*$, $i = 1, 2, \ldots, N-1$, where δ_* is a positive constant independent of both

ε and N. Firstly, for $i = 1, \dots, J-1$, we have

$$\begin{aligned}\tilde{l}_i v_{i-1} + \tilde{d}_i v_i + \tilde{r}_i v_{i+1} &= \frac{\overline{h}_i}{H} c_i v_i - \frac{\varepsilon}{h_i} + \frac{\varepsilon}{h_{i+1}} + \frac{b_i \overline{h}_i}{h_{i+1}} \\ &\geqslant -\left(\frac{\varepsilon}{h_i} - \frac{\varepsilon}{h_{i+1}}\right) + \frac{b_i}{2} + \frac{b_i h_i}{2h_{i+1}} \\ &= S_i \geqslant \delta > 0\,,\end{aligned}$$

by Lemma 2. Secondly, when $i = J$,

$$\begin{aligned}\tilde{l}_J v_{J-1} + \tilde{d}_J v_J + \tilde{r}_J v_{J+1} &= \frac{\overline{h}_J}{H} c_J v_J + \tilde{l}_J H - \tilde{r}_J \left(H + \frac{\lambda\rho}{1+\rho}\right) \\ &\geqslant -\tilde{r}_J \frac{\lambda\rho}{1+\rho} + (\tilde{l}_J - \tilde{r}_J)\, H \\ &\geqslant \left(\frac{2\varepsilon + b_J(h_J + H)}{2H^2}\right) \frac{\lambda\beta H}{\varepsilon + \beta H} - \frac{\varepsilon}{h_J} + \frac{b_J}{2} \\ &\geqslant \frac{\lambda\beta}{2H} - \frac{\varepsilon}{h_J} + \frac{\beta}{2} \geqslant \frac{\beta}{2}\,,\end{aligned}$$

provided

$$\frac{\lambda\beta}{2H} - \frac{\varepsilon}{h_J} \geqslant 0\,.$$

We bound h_J from below as follows:

$$h_J = \varepsilon \int_{t_{J-1}}^{t_J} \phi_B'(s)ds \geqslant \frac{\varepsilon}{N} \phi_B'(t_{J-1}) = \frac{\varepsilon}{N} \frac{2(1-\varepsilon)}{1 - 2(1-\varepsilon)t_{J-1}} \geqslant \frac{2\varepsilon(1-\varepsilon^*)}{N}\,,$$

where $0 < \varepsilon \leqslant \varepsilon^* < 1$. Therefore,

$$\begin{aligned}\frac{\lambda\beta}{2H} - \frac{\varepsilon}{h_J} \geqslant \frac{\lambda\beta}{2H} - \frac{\varepsilon N}{2\varepsilon(1-\varepsilon^*)} &= \frac{\lambda\beta N}{4} - \frac{N}{2(1-\varepsilon^*)} \\ &\geqslant N\left(\frac{\lambda\beta}{4} - \frac{1}{2(1-\varepsilon^*)}\right) \geqslant 0\,,\end{aligned}$$

where we use $N^{-1} \leqslant H \leqslant 2N^{-1}$, and we choose $\lambda \geqslant \frac{2}{\beta(1-\varepsilon^*)}$. Finally, when $i = J+1, \dots, N-1$, we have

$$\begin{aligned}\tilde{l}_i v_{i-1} + \tilde{d}_i v_i + \tilde{r}_i v_{i+1} &= c_i v_i + \tilde{l}_i H - \tilde{r}_i H + \tilde{l}_i \left[\frac{\lambda}{(1+\rho)^{i-1-J}} - \frac{\lambda}{(1+\rho)^{i-J}}\right] \\ &\quad + \tilde{r}_i \left[\frac{\lambda}{(1+\rho)^{i+1-J}} - \frac{\lambda}{(1+\rho)^{i-J}}\right] \\ &\geqslant b_i + \frac{\lambda}{(1+\rho)^{i+1-J}} \left[\tilde{r}_i - \tilde{r}_i(1+\rho) + \tilde{l}_i(1+\rho)^2 \right. \\ &\quad \left. - \tilde{l}_i(1+\rho)\right] \\ &= b_i + \frac{\lambda\rho}{(1+\rho)^{i+1-J}} \left[\tilde{l}_i - \tilde{r}_i + \tilde{l}_i\rho\right] \\ &\geqslant \beta + \frac{\lambda\rho}{(1+\rho)^{i+1-J}} \left[\frac{b_i}{H} - \frac{\beta}{H}\right] \geqslant \beta\,.\end{aligned}$$

By choosing $\delta_* = \min\{\delta, \beta/2\}$ we complete the proof. ♠

4 ε-uniform consistency and convergence

Let $\tau_i[g] = \mathcal{L}^N g_i - (\mathcal{L}g)_i$, $i = 1, 2, \dots, N-1$, for any $C^2(I)$-function g. In particular, $\tau_i[u]$ is the truncation error of the finite-difference operator $\mathcal{L}^N$ and

$$\tau_i[u] = \mathcal{L}^N u_i - \mathcal{L}^N U_i^N = \mathcal{L}^N (u - U^N)_i = [A_N(u^N - U^N)]_i\,, \qquad (16)$$

whereas the preconditioned consistency error is

$$\tilde{\tau}_i[u] = \begin{cases} \dfrac{\bar{h}_i}{H}\tau_i[u]\,, & i = 1, \dots, J, \\ \tau_i[u]\,, & i = J+1, \dots, N-1. \end{cases}$$

By Taylor's expansion

$$|\tau_i[u]| \leqslant C h_{i+1}(\varepsilon\|u'''\|_i + \|u''\|_i)\,, \qquad (17)$$

where $\|g\|_i := \max_{x_{i-1} \leqslant x \leqslant x_{i+1}} |g(x)|$ for any $g \in C(I)$.

Theorem 5. *Let $\beta > 2$. The preconditioned consistency error $\tilde{\tau}_i[u]$ is bounded uniformly in ε: $|\tilde{\tau}_i[u]| \leqslant CN^{-1}$, $i = 1, \ldots, N-1$.*

Proof: We use the decomposition (2) to get

$$\tilde{\tau}_i[u] = \tilde{\tau}_i[s] + \tilde{\tau}_i[y], \quad i = 1, \ldots, N-1,$$

and the estimates (3) to bound the terms on the right hand side separately. For the smooth part of the solution, $|\tilde{\tau}_i[s]| \leqslant CN^{-1}$, $1 \leqslant i \leqslant N-1$, due to (8) and $H \leqslant 2N^{-1}$. Then, for the singular component, we need to show that $|\tilde{\tau}_i[y]| \leqslant CN^{-1}$, $i = 1, \ldots, N-1$.

We divide the proof into cases regarding the indices i. For $i \geqslant J+1$, we apply (17) to y and use the derivative estimate (3). Then,

$$|\tilde{\tau}_i[y]| = |\tau_i[y]| \leqslant Ch_{i+1}\left(\varepsilon\|y'''\|_i + \|y''\|_i\right) \leqslant CN^{-1}\varepsilon^{-2}e^{-\beta x_J/\varepsilon} \leqslant CN^{-1}, \tag{18}$$

where use (11) and $\beta > 2$ in the last inequality.

For $i = J$, we have $|\tilde{\tau}_J[y]| = \frac{\bar{h}_J}{H}|\tau_J[y]| \leqslant C|\tau_J[y]|$, so we bound $|\tau_J[y]|$ directly by considering two cases: $\varepsilon \leqslant N^{-1}$ and $\varepsilon > N^{-1}$. First, when $i = J$ and $\varepsilon \leqslant N^{-1}$ we use the truncation error estimate in the form of $\tau_i[y] = \mathcal{L}^N y$, which is valid because of (4). Thus, we have

$$|\tau_i[y]| \leqslant P_i + Q_i + R_i, \quad P_i = \varepsilon|D''y_i|, \quad Q_i = b_i|D'y_i|, \quad \text{and} \quad R_i = c_i|y_i|.$$

We bound P_J from above as follows. Since $\bar{h}_J \geqslant h_{J+1}/2 \geqslant CN^{-1}$, we get $\bar{h}_J^{-1} \leqslant CN$ and invoking (10),

$$P_J \leqslant C\bar{h}_J^{-1}e^{-\beta x_{J-1}/\varepsilon} \leqslant CN\left(\varepsilon + 2N^{-1}\right)^{\beta} \leqslant CN^{-1}. \tag{19}$$

Analogous arguments can be applied to Q_J and R_J to imply that $|\tau_J[y]| \leqslant CN^{-1}$.

Second, when $i = J$ and $\varepsilon > N^{-1}$ we get that $h_J \leqslant C\varepsilon$ because of (8). Therefore, similarly to (18),

$$|\tau_J[y]| \leqslant CN^{-1}\varepsilon^{-2}e^{-\beta x_{J-1}/\varepsilon} \leqslant CN^{-1}\varepsilon^{-2}e^{-\beta x_J/\varepsilon} \leqslant CN^{-1}.$$

For $i \leqslant J-1$ we prove that

$$|\tilde{\tau}_i[u]| \leqslant CN^{-1} \quad \text{when} \begin{cases} i \leqslant J-2\,, \\ i = J-1 \quad \text{and} \quad h_J \leqslant \varepsilon\,, \\ i = J-1 \quad \text{and} \quad h_J > \varepsilon\,. \end{cases} \tag{20}$$

We first prove the first two cases of (20). For $i \leqslant J-1$ we have $h_i \leqslant \varepsilon$ because of (9) and $h_J \leqslant \varepsilon$ by the assumption. Hence,

$$\begin{aligned}|\tilde{\tau}_i[y]| &\leqslant C\frac{\bar{h}_i}{H}h_{i+1}\left(\varepsilon\|y'''\|_i + \|y''\|_i\right) \leqslant N^{-1}\left[\varepsilon\phi_B'(t_{i+1})\right]^2\left(\varepsilon^{-2}e^{-\beta x_{i-1}/\varepsilon}\right) \\ &\leqslant CN^{-1}\left[\phi_B'(t_{i+1})\right]^2 e^{-\beta x_{i+1}/\varepsilon} \leqslant CN^{-1}\left[1-2(1-\varepsilon)t_{i+1}\right]^{\beta-2} \leqslant CN^{-1}.\end{aligned}$$

Lastly, when $i = J-1$ and $h_J > \varepsilon$, this means that $\max\{\varepsilon, h_J\} = h_J$. Then, $\varepsilon \leqslant CN^{-1}$, again because of (8), and similarly to (19) we use (10) to get

$$\begin{aligned}|\tilde{\tau}_{J-1}[y]| &\leqslant C\frac{\bar{h}_{J-1}}{H}\left(P_{J-1} + Q_{J-1} + R_{J-1}\right) \\ &\leqslant C\frac{\bar{h}_{J-1}}{H}\left[\frac{1}{\bar{h}_{J-1}}\varepsilon \cdot 2\|y'\|_{J-1} + \frac{1}{h_J}\|y\|_{J-1} + e^{-\beta x_{J-2}/\varepsilon}\right] \\ &\leqslant CNe^{-\beta x_{J-2}/\varepsilon} \leqslant CNe^{-\beta x_{J-1}/\varepsilon} \leqslant CN\left(\varepsilon + 2N^{-1}\right)^{\beta} \leqslant CN^{-1}.\end{aligned}$$

♠

Combining Lemma 4 (ε-uniform stability) and Theorem 5 (ε-uniform consistency), we arrive the uniform convergence result.

Theorem 6. *On the Bakhvalov-type mesh defined in (6) and (7), the upwind difference scheme applied to the problem (1) is first-order uniformly convergent:* $\left|u_i - U_i^N\right| \leqslant CN^{-1}$, $0 \leqslant i \leqslant N$.

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