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Domain Coloring and the Argument Principle

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Domain Coloring and the Argument Principle

Abstract: The *domain-coloring algorithm* allows us to visualize complex-valued functions on the plane in a single image—an alternative to before-and-after mapping diagrams. It helps us see when a function is analytic and aids in understanding contour integrals. The culmination of this article is a visual discovery and subsequent proof of the argument principle, which relates the count of poles and zeros of a meromorphic function inside a contour to the accumulated change in argument of the function around the contour. Throughout, I offer connections to standard learning goals of courses in complex variables.

Keywords: complex variables, argument principle, domain coloring, teaching, contour integrals

1 INTRODUCTION

Most treatments of complex variables use only one tool to visualize complex-valued functions on the plane: the mapping concept, where before-and-after diagrams show what happens to a region of the complex plane when the given function acts on it. I certainly do not propose doing away with the mapping concept, but will argue that domain coloring [5] is a better place to begin. This alternative approach gets around the need to keep two images—*before* and *after*—in mind, though it's true one must recall certain coloring conventions; also, it is more closely

tied to what students have seen in functions of one and two variables: a diagram that shows the output values assigned to inputs in a domain.

First among the assets of the domain-coloring approach is the way it allows us to understand the meaning of the complex derivative—an adaptation of Needham’s *amplitwist* concept [8], but without the baggage of functions as mappings. Next, a sequence of graded examples bring us to guess the truth of the argument principle, one of the more sophisticated theorems typically presented in a first course on complex variables. To lead toward the proof of the theorem, I explain how domain coloring can help us go beyond a calculational approach (or complement one) to understand the value of a fundamental contour integral: the $2\pi i$ we get by integrating $1/z$ around the unit circle. (The residue theorem is easily understood from just this one family of computations.) We prove the argument principle and conclude by showing some domain colorings transferred to the Riemann sphere by stereographic projection.

Throughout, I focus on aspects directly connected to visualization and therefore skip over certain technical material, some of it easy and some more sophisticated. For instance, I assume without proof that a meromorphic function has a certain series expansion in a neighborhood of a pole. The standard reasons why we can deform contours of analytic functions are not aided by domain coloring, so I just deform them without comment. Similarly, I tacitly break up a contour into smaller pieces and add the results. This leaves some work for the reader in adapting these materials for a course. Still, I have tried to include relevant information, even reviewing the definition of the complex derivative. A student reader should be able to follow the ideas here, with reference to some of the standard texts. [1, 2, 7]

2 THE DOMAIN-COLORING ALGORITHM

The concept of *domain-coloring algorithm* can certainly be learned elsewhere [4, 5, 6], but we explain the concept to make this article self-contained. This technique for visualizing a function $f(z)$ of one complex variable involves two steps:

1. Assign a unique color to every point in the complex plane. For this paper, I do this by first following an artist's color wheel to assign pure hues around the unit circle, with red at the point 1, and then fading to black at the origin, white at infinity.
2. Color the domain of f by painting the location z in the plane with the color of the value of the function $f(z)$.

The first step might be called “specifying a color wheel” and the choices made in this step affect the execution of the second step dramatically [6]. Note that the coloring described here differs from others used in the literature in placing the color black instead of white at the center.

Figure 1 shows our color wheel. I believe that having distinct hue sectors and those rings to mark discrete levels of fading from black to white are more helpful than a gradual fade; they give us something to track visually when we execute the second step. This coloring lives on the domain where the absolute value of z is less than 2, which we call Ω_2 , and this means that complex numbers outside the domain will be lumped together and colored white.

Neither step of the domain-coloring algorithm can be accomplished perfectly, so domain coloring inherently involves approximation. In Figure 1, there are many pixels painted with a pure red color, so the assignment is not one-to-one. And when we apply step two, we can only paint a discrete set of pixels in the domain of f , limited by whatever image resolution we choose. Despite these imperfections, domain coloring

earns its keep as a visualization technique, as I will explain.

Figure 1. One way to assign colors to points in the complex plane. This specification of color wheel will be used in all domain colorings in this paper.

Before we go on, however, I should admit an obvious disadvantage of this technique: Not everyone sees color the same way. The most common variation is for people to see the colors at the points labeled R for red and G for green as being similar. Indeed, readers of the hard copy of this journal are probably seeing the images in grayscale and so need guidance. In some diagrams, I will include the letters, but they become cumbersome as the diagrams become more complicated, so I will explain how the shapes are intended as a guide. For now, suffice it to say that the three labeled primary colors—red, green, and blue—are favored for their role in RGB color projection on the screens at which we have become accustomed to stare.

The biological explanation for variations in color vision is far from simple, but indeed the physical frequencies of light that people refer to as “red” and “green” are closer together than are “green” and “blue.” It seems to be an accident of human biology that many of us see those three colors as equally different from one another. My pet theory is that

humans gained by being able to distinguish green leaves from red fruit, even though the colors are relatively close on the physical spectrum.

So the letters in Figure 1 are meant to aid people who see color differently, as well as anyone reading in grayscale. In addition, instead of a steady progression of hues equally spaced around the unit circle, I have included a narrow wedge enclosing the complex number i . This creates an asymmetry, which I'll use later to help us find the progression of hues counterclockwise around the color wheel. So, if you are a person with whom the names *red*, *green*, and *blue* do not resonate, try to learn to see the progression from a red sector of 60 degrees, through a smaller sector (orange) to a smallest sector (yellow), and know that this is the counterclockwise trip around the first quarter of the unit circle of complex numbers.

Having established a color wheel, we already have our first domain coloring: Figure 1 is already the domain coloring of the function $f(z) = z$ on Ω_2 , which, we recall, is our name for the domain where the absolute value of z is less than 2. One down.

Figure 2 shows another easy example. This is the complex squaring function, and understanding is enhanced by the polar formulas

$$z^2 = (x + iy)^2 = (re^{i\theta})^2 = r^2 e^{i2\theta}.$$

The domain coloring shows what happens when we square the radius and double the angle. Numbers inside the unit circle are painted with darker colors than in Figure 1 because the square of a number with radius less than 1 is smaller than the original radius. Similarly, numbers outside the unit circle have squares that are more distant from the origin, making the colors in the domain-coloring diagram lighter. Whereas, in our original color wheel, only numbers outside the circle of radius 2 were colored white, now the whole outside of the circle of radius $\sqrt{2}$ is colored white.

Figure 2. Left: The domain coloring of the squaring function shows that this operation doubles the angle and squares the radius. Right: A domain coloring of a polynomial of degree six.

This domain coloring of the squaring function gives a first hint of why this is such a powerful technique to visualize the argument principle: This function has the prototypical double zero, and the colors indeed “go around twice” as we travel around the unit circle in the domain. The narrow wedge of yellow about the point i is helpful here for visual tracking: We see it twice as we travel the unit circle.

Even if we decide to travel an exceptionally large circle around the origin, one so large that the imperfections of our particular effort to color the complex plane leads us to see only washed-out whiteness, it is plausible that our circuit travels twice around the color wheel, however light the hues may be.

On the right in Figure 2 is a domain coloring of a sixth degree polynomial,

$$f(z) = z^2 (z - (2 + 2i)) (z - (-1/2 + 2i)) (z - (-2 + i)) (z - (-3 - i)).$$

In this factored form, you can see where the zeros of the polynomial are, including the double zero at the origin. Find them in the figure as the

points painted black. For this image, I treated the letters R, G, and B in Figure 1 as if they were part of the color wheel, so they become turned and distorted as the function demands. An alternative is to compute with colors alone and use software to stick letters in afterward, as I did for the squaring function—observe that the shapes of the letters are undistorted—but this would be cumbersome for the sixth degree polynomial.

The zeros of this sixth degree polynomial were chosen to echo a diagram from [5], which might have been given the caption, “The Scream.”

Perhaps this is a good time to explain that my images are prepared using a C++ program designed for domain coloring. Others have implemented this idea in a variety of platforms, particularly Mathematica. I am happy to share my code, which is open source, but do not promise to support it. (Link to code will go here, either at author’s website or Primus site.)

3 DOMAIN COLORING AND FUNDAMENTALS OF COMPLEX ANALYSIS

If the concept of domain coloring is introduced early in a course, it can support discussions of the most basic questions: Which functions are analytic? What is the complex derivative, when it exists?

The first large task of any teacher of complex variables, after establishing the easy algebra of complex numbers, with their rectangular and polar forms, is to convey what is meant when we say that “the function $f(z)$ is *analytic* (or *holomorphic*) on a domain Ω .” In *Visual Complex Analysis*, Needham [8] does an excellent job with this by introducing the concept of “amplitwist.” However, the amplitwist, measuring how much a function amplifies and turns the plane, is inherently tied to the mapping concept of complex functions on the plane. Let us modify that

idea, while staying close to its essence: that the defining property of complex analytic functions is the existence of a linear approximation at every point.

First, I ask you to mentally construct the domain coloring of a typical linear function, which I will write as $f(z) = cz = se^{i\alpha}z$, where the polar form of the constant c is $se^{i\alpha}$, for real s and α . If you are used to thinking of f as a mapping, you will imagine that this function scales by a factor of s and turns by an angle α . That isn't wrong, but it expressed the idea in terms of mapping, not in terms of values of the function f . For the domain coloring of this f , we would see light colors brought in closer to the origin if $s > 1$ and dark colors pushed out from the middle if $s < 1$. If $\alpha > 0$, the hues at points $x + i0$, for $x > 0$, are those from further counterclockwise around the circle, such as orange and yellow, depending on how large α is. This might seem backward if you are too used to the mapping paradigm, but it matches exactly what the values of f are at the points you are viewing.

In any case, the domain coloring of a linear function f will look like a scaled and rotated color wheel, with the same scaling in all directions. This same scaling in all directions means that the function is *conformal*, a term whose definition is closely tied to the mapping concept: A conformal mapping is one that preserves angles but not necessarily distances. Evidence of this conformality can be seen in Figure 2, where the intersections of curves bounding different hues and curves bounding different brightnesses are always perpendicular. (As a technical aside, some people have told me that they think of the domain coloring of f as representing a mapping diagram of the *inverse function* of f , though that is sometimes multiple-valued and not an actual function. This inverse-function view tells us that the sector boundaries meeting the radial borders at right angles in a domain coloring is a sign of conformality.

We will not emphasize this point.)

Let's review the definition that connects differentiability to linear approximations. The *derivative* of the function $f(z)$ at the point $z = a$ is defined by

$$f'(a) = \lim_{z \rightarrow a} \frac{f(z) - f(a)}{z - a},$$

provided the limit exists. We should explicitly mention that the existence of this limit requires the value to be the same when z approaches a from any direction whatever. If $f'(a) = c$, then the numerator $f(z) - f(a)$ is approximately the function $c(z - a)$ (linear in the quantity $(z - a)$) when z is close to a .

To investigate this visually, we localize the function f to the point a as $f(z) - f(a)$ and compute a domain coloring: the image should look like a linear map, which is to say, a scaled and rotated color wheel.

Let's try this for the squaring function, using two points for localization, 1 and i . Since we probably already know that $(z^2)' = 2z$, it is fruitful to compare the appearance of the localized functions at the two given points with the domain colorings of functions $2(z - 1)$ and $2i(z - i)$. They indeed match what we see in Figure 3, as long as we look close to the points in question. Of course, this does not prove anything, but it shows how domain coloring supports understanding of the meaning of the derivative.

Since $(-1)^2$ is the same as 1^2 , the left half of the left-hand side of Figure 3 also helps us confirm that the derivative of z^2 at $z = -1$ is -2 . Similarly, in the lower part of the right-hand figure, we have visual evidence that the derivative of z^2 at $z = -i$ is $-2i$. The limit process required for proof is simple:

$$\lim_{z \rightarrow a} \frac{z^2 - a^2}{z - a} = \lim_{z \rightarrow a} z + a = 2a,$$

so we get an excellent match between computation and visualization.

Figure 3. Left: The domain coloring of $z^2 - 1^2$ looks linear near the point $z = 1$. Right: Similarly, near $z = i$, the localized function $z^2 - i^2$ is approximately $2i(z - i)$.

For a contrasting example, consider

$$f(z) = \frac{3}{2} (z - z\bar{z}i),$$

whose domain coloring is shown in Figure 4. (The factor of $3/2$ was chosen to improve the visual appearance of the diagram.)

I leave it to you to prove that the complex derivative of this function exists only at the point $z = 0$ and that $f'(0) = 3/2$. Indeed, the definition of f suggests that it is approximately equal to $3z/2$ when z is small. Why does the function look so similar near $z = i$, the other zero of this function? We notice that the colors go around backward near that point, and so compute the derivative of f , not with respect to z but with respect to $\bar{z}$ at that point:

$$\frac{\partial}{\partial \bar{z}} f(z)|_{z=i} = \lim_{z \rightarrow i} \frac{f(z) - f(i)}{\bar{z} - \bar{i}} = \lim_{z \rightarrow i} -\frac{3}{2} iz = \frac{3}{2}.$$

Our function is not analytic at that point, but anti-analytic! At other points, we can notice the nonconformal nature of this function in the various non-perpendicular intersections. An example like this one can

Figure 4. Left: The domain coloring of $f(z) = 1.5(z - z\bar{z}i)$, looks linear near the point $z = 0$, but the colors cycle in the opposite direction at the point $z = i$.

go far to show students the meaning of differentiability with respect to z .

4 ANALYTIC FUNCTIONS

After the definition of the complex derivative is digested, the next important message of the teacher of complex variables is the surprising strength of the hypothesis that a function is *analytic*, by which we mean that the function is differentiable not just at a point, but in some neighborhood. Higher derivatives of any order follow from the existence of a single one, as long as that derivative exists in a neighborhood. It's a long story, which I have found few ways to illustrate with domain coloring, but one highlight is that a function $f(z)$ that is analytic in a punctured disk $0 < |z - z_0| < R$ can be expressed as the sum of a *Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n, \quad (1)$$

where the series converges in some punctured disk of radius possibly smaller than R .

We identify three cases of interest: If $a_n = 0$ for all $n < 0$, then f is analytic in a disk. If there is some positive integer M so that $a_{-M} \neq 0$ but $a_n = 0$ for $n < -M$, we say that f has a pole of order M at z_0 . These first two cases are considered the nice ones. If there are nonzero coefficients with arbitrarily large negative indices, then f is said to have an *essential singularity* at z_0 .

The function

$$f(z) = \frac{1}{2}e^{1/z} = \frac{1}{2}e^{x/(x^2+y^2)}e^{iy/(x^2+y^2)}$$

illustrates the prototypical essential singularity at the origin. It's domain coloring appears in Figure 5. (Again, the factor of $1/2$ was included just to bring the colors more into the range where we can see them.) With a little imagination, one can see evidence for the Picard Theorem, which says that f must assume every value but one (0 in this case) in every neighborhood of the origin!

Figure 5. The function $f(z) = 0.5e^{1/z}$ has an essential singularity at the origin.

The examples that occupy the remainder of the paper are far nicer

than this one. These will be *meromorphic* functions, whose only singularities in their domain of definition are poles (of finite order). In domain colorings of these functions, we will find evidence for the argument principle.

5 THE ARGUMENT PRINCIPLE

We build on something we noticed back in Figure 2: In the domain coloring of the squaring function, when we make any counterclockwise closed circuit around the origin, the colors cycle twice. The function z^2 only has one zero in the plane, but we call it a *zero of multiplicity two* and in a count of zeros, it counts twice.

Look back at the right-hand image in Figure 2, the sixth degree polynomial. Follow counterclockwise around the central dark region, the one that might look like a screaming face. The colors go RGB-RBG-RBG-RBG—four times around. And there are exactly four zeros inside that path, counting the one at the origin twice.

Figure 6 shows the domain coloring of the rational function

$$f(z) = \frac{1 + z^3 + z^6/10}{1 + iz^3 + z^6/10}.$$

The polynomial in the numerator has six distinct roots and these appear in the domain coloring as the six black dots, where the colors cycle around once in their proper order. Similarly, each of the distinct zeroes of the denominator appears as a white dots. Which way do the colors cycle around those? They go in the opposite direction! A quick thought experiment tells us why: The reciprocal function, $f(z) = 1/z$, also called *complex inversion*, can be written in polar coordinates as

$$f(re^{i\theta}) = \frac{1}{r}e^{-i\theta}. \quad (2)$$

The domain coloring of this function will show the colors progressing, as we look counterclockwise, from red to blue to green. The yellow

Figure 6. This rational function has six zeros and six poles. The values at 0 and infinity are both 1.

sector we added as a visual cue will be clockwise from red in the domain coloring, not counterclockwise as we usually find it. (For a visual display, look ahead to Figure 9.)

This is why we see the reverse color cycling in Figure 6: Each of the simple zeros of the denominator is a simple pole (order 1) of the function, leading to behavior like what we see in $1/z$ near the origin.

We pause to explain the 3-fold symmetry of this function, which is caused by all powers of z being multiples of 3. It's an easy computation to check that

$$f(\omega_3 z) = f(z), \text{ where } \omega_3 = e^{2\pi i/3}.$$

Along with the fact that multiplication by ω_3 turns the plane by 120° , this equation shows why the domain coloring has 3-fold symmetry. I thought the symmetry was appealing, but it makes a big difference when we depict the function on the Riemann sphere in the last section of this article.

To build to a more complicated form of the argument principle, traverse the simple closed curves in Figure 7.

When we trace counterclockwise around the one on the left, starting in the middle of the figure near the letter R, we count RGB before arriving at the dark (unlabeled) red, then pass through GB before returning to our starting point. That's twice around and there are two zeros inside. This is just what we counted for the squaring function: twice around for a doubly-counted zero at the origin. The change in argument seems to be the same whether the zeros are distinct or combined into a double zero.

Figure 7. Two contours drawn on the domain coloring of the same function. Compare the total change in argument with the count of zeros and poles inside.

Now trace the right-hand curve counterclockwise, starting near the R in the center. We move toward purple, but then back to red. No change in argument yet. We spend a fair amount of time in the dark red area, but then coming around the top left of the curve, we quickly pass G and B, completing one cycle (forward) through the colors. The total contribution to the change in argument by *two zeros* inside the curve has been canceled by the reverse change caused by the *one pole* inside.

Try it with other curves on the same diagram, and on the other diagrams given, and on your own domain colorings. Just don't pass

through a pole or a zero. No matter what meromorphic function you choose, a simple closed curve that avoids poles and zeros must obey the argument principle, which we state informally as:

The total change in argument around a simple closed curve, measured in multiples of 2π , must equal the number of zeros inside minus the number of poles inside (counting multiplicities).

Let's view one more example, more for beauty than for discovery. The function shown in Figure 8 is

$$f(z) = 10 + z^5 + z^{-5} + 0.03iz^{10} + 0.03iz^{-10}.$$

Based on our discussion of 3-fold symmetry, the reader should have no trouble seeing why this one has 5-fold rotational symmetry about the origin. It also has a different, more interesting symmetry: $f(1/z) = f(z)$. This function is invariant under complex inversion, which gives a nice rhythm to its domain coloring. Spend a moment looking for features in the figure to confirm that complex inversion “inverts the radius and negates the angle,” as in (2).

Figure 8. A function with 5-fold and inversive symmetry. Count its zeros.

When z is large, the dominant term is the z^{10} one, and this produces

a total change in argument of 10 times around. The only pole is the one of order 10 at the origin, so the number of zeros must be the integer N that satisfies $10 = N - 10$, so there must be 20 zeros, and we can see them in the diagram.

The new text *Explorations in Complex Analysis* [3] guides students toward the argument principle using software to illustrate the winding number of various circles under complex mapping, with instructions to apply mappings to specific contours. The explorations are well chosen, but I find the visual evidence for the argument principle less compelling than what we see with domain colorings.

6 PROVING THE ARGUMENT PRINCIPLE WITH CONTOUR INTEGRALS

The argument principle is easy to prove, once we have access to facts about contour integrals. The two ingredients are 1) our ability to break up contour integrals into pieces, so that one feature is enclosed at a time, and add the results and 2) the value of a fundamental integral. Keeping with our intention to give visual support to plausibility arguments, we work at an intuitive level. To make this rigorous, we also need to know that functions have certain series expansions, as I will explain.

I assume that the reader is comfortable thinking of contour integrals as limits of approximating sums:

$$\int_C f(z) dz \approx \sum_{k=1}^n f(z_k)(z_k - z_{k-1}),$$

where the points $z_0, z_1, \dots, z_n = z_0$ divide the simple closed curve C into pieces that are close to being line segments. This interpretation makes it easy to understand why

$$\int_{|z|=1} 1 dz = 0.$$

Just think of the terms $z_k - z_{k-1}$ as being so many vectors that, when their tails are placed at the origin, will all add up to 0.

For those uncomfortable with the intuitive approach, we offer the standard technique of parametrization: Parametrize the contour $|z| = 1$ by $z(t) = e^{it}$, so that $dz = ie^{it}dt$, and compute

$$\int_{|z|=1} 1 \, dz = \int_0^{2\pi} 1ie^{it}dt = e^{it} \Big|_0^{2\pi} = 0.$$

Perhaps the most important contour integral for students of complex analysis to understand and remember is

$$\int_{|z|=1} \frac{1}{z} \, dz = 2\pi i.$$

This computational method is easy to apply (try it!), but the visual approach has a lot to offer. Figure 9 helps us see *why* the value turns out the way it does.

Figure 9. The integral of $1/z$ around the unit circle is $2\pi i$.

I've drawn only three arrows to represent typical segments $z_k - z_{k-1}$. Start with the left-most of these, the downward-pointing arrow at $z = -1$. The value of $1/z$ at this point is -1 , so we turn that arrow around and it points upward. Similarly, the arrow in the lower right-hand corner

is multiplied by the value of $1/z$, which is the color *orange*, somewhere around $e^{i\pi/4}$, again turning the arrow upward. The arrow in the upper right-hand corner is likewise turned vertically, as indeed are all the arrows in the approximating sum. The arrows all line up and add to a sum of $2\pi i$, since the circumference of the circle is 2π . (This explanation is borrowed from Needham [8], but modified to use domain coloring rather than mapping.)

Expanding or contracting the circle changes the value not at all! The length of the arrows would grow by a factor R which would cancel with the $1/R$ in the value of the function $1/z$. This fact gives a first glimpse of our ability to move contours without changing the values of integrals.

If we replace $1/z$ with *any other integer power* of z , positive or negative, the arrows all point in different directions and cancel out, just as they did when we integrated 1 around the closed contour. This is the practical understanding at the heart of the Residue Theorem, provided we have the theoretical capacity to replace a function $f(z)$ by a Laurent series $\sum_{n=-m}^{\infty} a_n z^n$, that converges in a neighborhood of the origin. The contour integral of f around a small circle containing the origin picks up only the “residue,” $2\pi i a_{-1}$.

With these facts in mind, we are ready to prove the argument principle.

We assume familiarity with the function $\text{Log}(z)$, the principal branch of the logarithm function, which can be defined by

$$\text{Log}(z) = \text{Log}(re^{i\theta}) = \ln(r) + i\theta \text{ where } -\pi < \theta \leq \pi.$$

The imaginary part of $\text{Log}(z)$ conveniently counts up changes in argument. What we have described in visually counting the change of argument around a contour can be captured technically by the total

change in $\text{Log}(f(z))$. A quick check shows that the contour integral

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz$$

is exactly what we need to count the number of times the argument of $f(z)$ cycles (forward) around the circle as we travel C counterclockwise.

Now suppose that C is a simple counterclockwise closed curve that misses all poles and zeros of a meromorphic function f . We split C up into contours so that each one surrounds only one pole or one zero, labeling the contours that surround zeros as $Z_1, Z_2, \dots$ and those that surround poles as $P_1, P_2, \dots$.

If f has a zero of order m_k at the point z_k and Z_k is the contour that encloses it, we write

$$f(z) = (z - z_k)^{m_k} g_k(z),$$

where $g_k(z)$ is an analytic function that does not vanish at z_k . Compute

$$\frac{1}{2\pi i} \int_{Z_k} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \int_{Z_k} \frac{m_k}{z - z_k} + \frac{g'_k}{g_k} dz = m_k,$$

where the last step uses our fundamental computation for the function $1/z$ and the fact that g'_k/g_k is analytic.

The calculations for poles are similar: If f has a pole of order m_k at the point z_k surrounded by contour P_k , then

$$f(z) = (z - z_k)^{-m_k} g_k(z),$$

for non-vanishing, analytic g , and

$$\frac{1}{2\pi i} \int_{P_k} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \int_{P_k} \frac{-m_k}{z - z_k} + \frac{g'_k}{g_k} dz = -m_k.$$

Combining the steps, we see why

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = \text{number of zeroes inside } C - \text{number of poles inside } C,$$

where the numbers are counted with multiplicity.

7 THE RIEMANN SPHERE

Let's close with an additional visualization, one that is usually mentioned in courses on complex variables, but not previously—to my knowledge—combined with domain coloring. The completion of the complex plane $\mathbb{C} \cup \{\infty\}$ is identified with the sphere S^2 , called the *Riemann sphere*, by stereographic projection. The exact projections vary slightly, but I will use what I think is the standard one, where the north pole of S^2 corresponds to the point ∞ and the equator to the unit circle.

Under this correspondence, functions on the plane become functions on the sphere and conversely, modulo possible problems at infinity. (For instance, the function e^z has an essential singularity at infinity. Its domain coloring near infinity would be like Figure 5!) There is no such difficulty with the functions z and $1/z$. Domain colorings for these functions on the sphere appear in Figure 10. In this view of the sphere, we are looking down on the north pole, so this is a good way to see what's going on at infinity. On the left, we see white at the north pole; on the left, that's black, since the function $1/z$ has a zero at ∞ .

Figure 10. Domain colorings of z and $1/z$ transferred to the Riemann sphere.

Alas, there's something wrong with these pictures. Imagine looking

from below and moving from red (front center) toward the labeled green area. This moves the colors in the direction we have called forward, and yet that is a clockwise, not counterclockwise circuit around the south pole. The problem is that stereographic projection takes the positive orientation of the plane to the *inward* orientation of the sphere, so our conventions for domain coloring are backward, unless we can imagine ourselves to be inside the sphere. Perhaps the next generation's virtual reality goggles will permit them to acquire that view with ease.

Domain colorings of the two more complicated rational functions we studied earlier appear in Figure 11. Modulo the adjustment of reversing orientation, they depict the functions quite nicely. This is why we selected functions with rotational symmetry: We get a nice view of what's going on from seeing only a portion of the sphere!

Figure 11. Domain colorings of two rational functions (previous examples) transferred to the Riemann sphere. On the left is the function with six poles and six zeros; on the right is the one with 5-fold and inversive symmetry.

8 CONCLUSION

Visual presentations of information are more and more a part of the world our students live in. I hope that the domain-coloring algorithm can become a standard tool for conveying the essence of complex-valued functions on the plane. It is easy to implement and is so flexible that students with a variety of visual abilities can find what they need to connect with the geometric side of complex analysis.

The striking difference between domain colorings of analytic and not-analytic functions will surely create more lasting memories than the most concise exposition of the Cauchy-Riemann equations. And it's hard to argue with the argument principle, when it is so easily observed in domain-coloring diagrams.

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BIOGRAPHICAL SKETCHES¹

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