On the Laplacian

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Marcus [2]).

3) The theorem can be used to generate identities valid in *any* ring. For example,

$$r_1^2 + \cdots + r_n^2 = (r_1 + \cdots + r_n)^2 - 2(r_1 r_2 + \cdots + r_{n-1} r_n);$$

$$r_1^3 + \cdots + r_n^3 = (r_1 + \cdots + r_n)^3 - 3(r_1 + \cdots + r_n)(r_1 r_2 + \cdots + r_{n-1} r_n) + 3(r_1 r_2 r_3 + \cdots + r_{n-2} r_{n-1} r_n).$$

The above results can be extended to the case $k = 0$ or k a negative integer, when a_0 is nonzero. This would involve taking the inverse of the companion matrix. (The latter is easily obtained, however, by any of several methods, e.g., matrix adjoints or the Cayley-Hamilton Theorem. Also, it is not too hard to see that this inverse closely resembles the original companion matrix.)

References

[1] I. N. Herstein, Topics in Algebra, Xerox College Publishing, 1964.

[2] D. A. Marcus, Number Fields, Springer-Verlag, 1977.

[3] B. L. van der Waerden, Frederick Ungar Publishing Company, 1953.

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In various applied mathematics courses one appearance of the Laplacian operator ∇^2 is in the study of heat distributions. If u is a heat distribution in space, then $\nabla^2 u = 0$ if and only if u is a steady-state distribution, one that could be maintained indefinitely inside a box with suitable boundary conditions.

In rectangular coordinates the Laplacian of u is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

Careful use of the chain rule gives

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

and

$$\nabla^2 u = \frac{\partial^2 u}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2 \sin^2\phi}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\cot\phi}{\rho^2}\frac{\partial u}{\partial \phi}$$

as the correct formulas for the Laplacian in cylindrical and spherical coordinates [1].

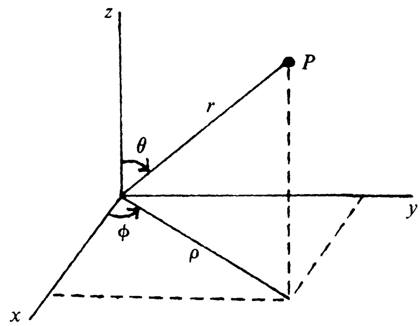

FIGURE 1

Since problems in two dimensions are often easier to solve than those in space, it is desirable to recover a two-dimensional Laplacian from the formulas above. In the first two instances this is straightforward: differentiations with respect to z may be ignored, leaving

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$$

and

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

as the correct formulas for the Laplacian in rectangular and polar coordinates in the plane. Students readily accept the physical reason justifying this procedure: a steady-state heat distribution on a thin metal plate will also give a steady-state distribution when extended to space in such a way that it is independent of z. For example, $u(x, y)=xy$ could equally well describe the steady-state heat profile of a thin disk or of a cylinder.

A problem commonly arises, however, when students attempt the same reduction of the Laplacian in spherical coordinates. At first it seems reasonable to ignore differentiations with respect to ϕ, set $\phi=\pi/2$, and replace ρ with r. This gives

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2},$$

apparently contradicting the formula above. Since the difference is a matter of a factor of 2, students may shrug away the contradiction as a misprint.

What is responsible for the incorrect coefficient of $\partial u/\partial r$? Students must examine more carefully what is entailed in ignoring differentiations with respect to ϕ. To do so is to pretend that one's two-dimensional heat distribution is actually a distribution in space which is independent of ϕ. The resulting heat distribution would be constant on the longitudes of the unit sphere and singular at the North Pole. Certainly the question of heat equilibrium for the two distributions is not the same. A steady-state distribution on a plate would yield quite an unsteady distribution when extended to space in this manner.

Having dispensed with the naive approach, we must turn to the chain rule for the correct computation. If $u(r, \theta)$ is a function in the plane, extend it to a function in space by supposing it to be independent of z. The Laplacian of the extended distribution will also be independent of z

and, when restricted to the plane, will give the correct planar Laplacian for the original function u. Applying the Laplacian to u amounts to computing the spherical partial derivatives of u in terms of the cylindrical derivatives. For instance, since $r = \rho \sin\phi$,

$$\frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial r}\rho\cos\phi + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \phi}$$

but $\partial u/\partial z$ is zero. Differentiating again shows that

$$\frac{\partial^2 u}{\partial \phi^2} = \frac{\partial^2 u}{\partial r^2}\rho^2\cos^2\phi - \frac{\partial u}{\partial r}\rho\sin\phi.$$

But when this is restricted to the plane where $r = \rho$ and $\phi = \pi/2$, we find that

$$\frac{\partial^2 u}{\partial \phi^2} = -\frac{\partial u}{\partial r}r.$$

This term corrects the coefficient of $\partial u/\partial r$ which was wrong in the first approach. The other terms are easy to check; for instance, $\partial^2 u/\partial \rho^2$ really does translate to $\partial^2 u/\partial r^2$ when $\phi = \pi/2$.

Another explanation of the error in the naive approach is possible for students who understand that the Laplacian of a function is the divergence of its gradient vector field. If a planar function whose Laplacian is zero is extended to be independent of z, then the gradient of the resulting function is also independent of z and its divergence is zero. On the other hand, the same function extended to be independent of ϕ will typically pick up nonzero divergence in its gradient. A one-dimensional analogue illustrates this phenomenon in a setting where the calculations are simple: the function $f(x) = x$ has the unit vector $\vec{i}$ as its gradient and the divergence of this is zero. However, if f is extended to the plane so that θ is constant, the resulting function $v(r,\theta)$ has the unit vector field

$$\frac{x}{r}\vec{i} + \frac{y}{r}\vec{j}$$

as its gradient. The divergence of this field is $1/r$. By extending the function in a strange way, extra divergence may be introduced in its gradient vector field.

At a more sophisticated level this phenomenon could be used to contrast the push-forward of the Laplacian under two different projections from space to the plane. However, typical students working through this dilemma may learn to appreciate the chain rule and the need for care in changing coordinates.

Reference

[1] R. V. Churchill and J. W. Brown, Fourier Series and Boundary Value Problems, 4th ed., McGraw-Hill, New York, 1987.