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A Window on the Fifth Dimension

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WHAT'S INSIDE

4 .... A Window on the Fifth Dimension

10 .... Preparing Students for a Life in Math or Computer Science

14 .... An Intuitive Approach to the Borsuk-Ulam Theorem

20 .... Write About Mathematicians in Non-Major Courses

A Window on the Fifth Dimension

By Frank A. Farris

Is there enough mathematics in your home? What visual aids do you keep on hand for that inevitable moment when guests want to know why you spend your life on mathematics? Feeling a lack in this area, I commissioned glass artist Hans Schepker to produce a window — from the fifth dimension? — based on an image that came up in my research. It turned out splendidly, and you can see it on the cover of this issue of MAA FOCUS.

The story of the design involves complex functions, color as a visualization tool, and five-fold symmetry. I cannot tell it to every guest who walks in the door, but mathematicians will find it a strange combination of the familiar and the unlikely. Can this window make the fifth dimension transparent?

The Human Story

At the 2007 JMM in New Orleans, I met Hans Schepker, who was exhibiting his glass art. You may have seen his booth or visited his web site, http://glassgeometry.com. A rainbow pentagon design caught my eye and I asked whether he might be able to make me a custom window for my home. Hans was enthusiastic, but I did nothing about the project until I saw him again a year later at the JMM in San Diego.

It turned out that Hans was to spend several weeks that winter in my neighborhood, giving a geometry course at a Waldorf School, using glass art as the hook to interest students. We met at my house and talked about possibilities, settling on a computed image — approximately the one in Figure 1 —as the pattern for a window.

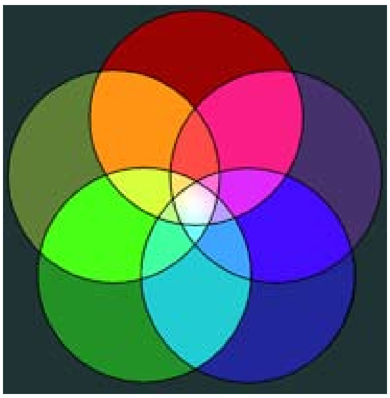

Figure 1: The original schematic design

The next, rather difficult, step was selecting the actual glass for the project. I spent a couple of hours with Hans at the Palo Alto showroom of Franciscan Glass — enough to learn the difference between pointing to a color on a screen and having a piece of glass of just that color.

A week or so later, I asked my college friend Kitty Kameon, who has taught color theory at the San Francisco Academy of Art University, to meet me at the Franciscan Glass *warehouse*, across the bay at the eastern foot of the Dumbarton Bridge. We spent more hours sorting through possibilities, which after a time ceased to appear finite. The goal was to implement certain rules from the mathematical original: Crossing one of the circular boundaries leads to a neighboring color; moving from the center outward should lead from white through pastels to vivid saturated colors and then to darkness at the outside. (The outer darkness was replaced by clear glass in order to help the window appear to float in its frame.)

In addition to finding *colors* for each of the four rings of five congruent shapes, we thought about the balance of textures. In some places, the window is not entirely symmetrical. We decided that mathematical precision could give way to aesthetic considerations, given that the glass choices were finite. The bluish sector was the easiest part; there were more than enough choices. We spent more time on the sector where greens become oranges and then reds. I especially like the gold crown at the top of the window. Overall I very much appreciate how Kitty's advice led to harmonious colors that honor the spirit of the mathematical design.

Once Hans had all this information, he faced the engineering challenge of making the window strong enough. A typical glass window has at least some straight lines — an obvious source of strength. This window has only curves. Despite the challenges of the unique design, Hans sent the finished window to me after only six weeks' work. It arrived in a custom-made plywood crate and has delighted me from my first view.

The Mathematical Story

The mathematics behind the design started with my desire to find a mathematically correct depiction of a color wheel, something I have been trying to do for years. My need for a good color wheel arose in the mid-90s, when I proposed a particular way to use color to depict complex-valued functions on the complex plane [**2**]. The first step is to color the complex plane in such a way that, theoretically at least, each point has a different color. I like to use the primary colors red, green, and blue to color the cube roots of unity, then fill in hues around the unit circle, and fade to white at 0 and black at infinity. To create a *domain coloring* for a function $f(z)$, at each z-value in the domain, paint the color corresponding to $f(z)$. More realistically, for the z-value of each *pixel* in the picture, set the color value of that pixel to $f(z)$.

For instance, using the rather crude color wheel on the bottom in Figure 2 to indicate the color scheme, I produced the domain coloring of a sixth-degree analytic polynomial beneath the color wheel. The five white areas indicate the zeroes of the polynomial, which might suggest that we are one zero short. Around the largest white area, note that the colors cycle twice around the edge: This is a *double* zero — just one example of the many phenomena revealed by domain colorings.

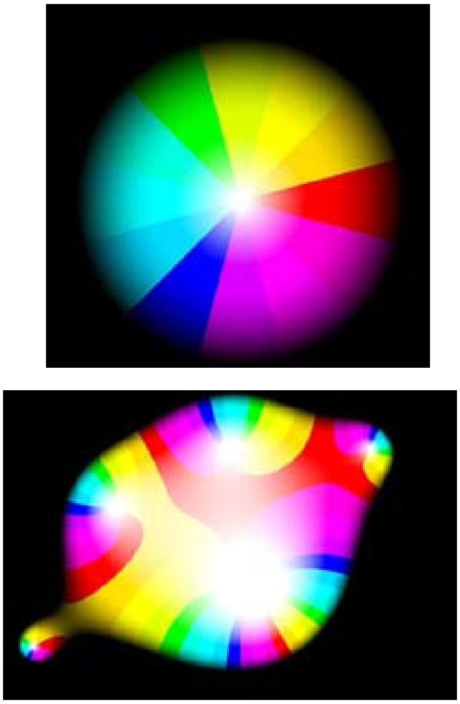

Figure 2: A choice of color wheel and the corresponding domain coloring of a sixth degree polynomial.

I have worked with several color wheels, each one a kludge. Then, in the summer of 2007, I prepared a talk for the MAA Silver and Gold Banquet at the San Jose MathFest. The new color wheel was mathematical!

Most of my computed images have been coded in a file format called PPM. It is rather an antique file format, but it is extremely simple and portable. For each pixel, the file lists three values from 0 to 255 to indicate how much red, green, and blue light should shine from that pixel. These are called RGB values and, independent of file format, most screens you view are being addressed via RGB values. The set of possible RGB values forms a rather obvious cube, with black at one corner, as (0,0,0) and white, as (255, 255, 255), at the opposite corner. This cube was the inspiration for my next steps.

New Idea: Use stereographic projection to map the complex plane to the unit sphere. Map the unit sphere inside the RGB cube, tilted so that the pole corresponding to 0 is nearest to (255, 255, 255) and the equator point that came from 1 is near the red corner. Then read off the colors.

It takes a little computation to get white in the right place, and to pursue my intention that 1 should be colored red. Here is the formula that takes complex numbers to a correctly-tilted sphere inside a cube: $z = u + iv \rightarrow (X, Y, Z) =$

$$\frac{1-u^2-v^2}{1+u^2+v^2}\frac{(1,1,1)}{\sqrt{3}} + \frac{2u}{1+u^2+v^2}\frac{(2,-1,-1)}{\sqrt{6}} + \frac{2v}{1+u^2+v^2}\frac{(0,1,-1)}{\sqrt{2}} \in [-1,1]^3$$

Readers may recognize that the coefficients of the vectors arise from stereographic projection. It is conceivable that someone might recognize the three vectors: They are unit eigenvectors of the cyclic permutation of three variables. The first one points along the main diagonal of the cube, toward the vertex that will represent white. The second and third span the plane of the celebrated regular hexagonal cross-section of the cube.

All that remains is to map this 2 x 2 x 2 cube to the color cube:

$$(X, Y, Z) \rightarrow \left(\frac{1+X}{2}, \frac{1+Z}{2}, \frac{1+Y}{2}\right).$$

This is so theoretically lovely that it should yield beautiful pictures. Unfortunately, this mathematical color wheel looks terribly bland.

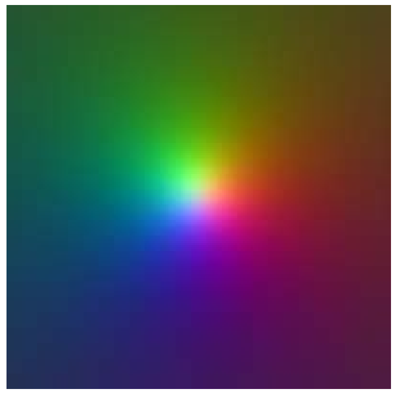

Figure 3: A color wheel derived by stereographic projection of the complex plane into the color cube.

This is where my pragmatic nature took over, happily leading me to unexpected consequences: I decided to take the cube root of each component of $(X, Y, Z) \in [-1,1]^3$. Why? This would drive points outward toward the edges of the cube; brights would become brighter, darks darker. Before you read on, ponder the resulting color wheel. What is Figure 4 a picture of?

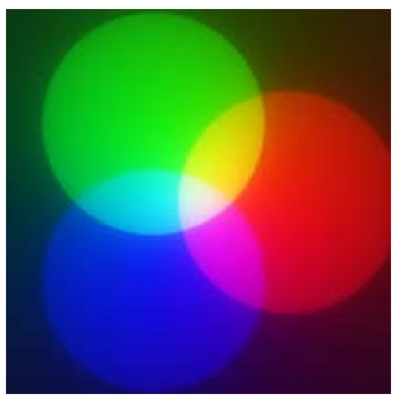

Figure 4: A color wheel derived from the previous by taking cube roots coordinate by coordinate, for no good reason.

If you are discretely minded, you might recognize the graph of a regular octahedron, viewed by stereographic projection. A continuously minded mathematician might focus on the circular boundaries, which are unstable sets of the cube root map, the traces of the coordinate planes. Let's explain each point of view.

Picture what the cube root map does to the interval $[-1,1]$. It moves positive values toward 1 and negative values toward -1; these points are stable attractors. The origin is unstable; points on either side, though they may be very close, are moved toward different endpoints. Now observe that I applied this map to each coordinate of a triple inside the cube. This moves points away from the coordinate planes and toward the vertices of the cube.

The reason we see eight regions of almost constant color is that all points in one of those regions have been pulled toward the same vertex of the cube, of which there are eight. The reason we see dividing circles is that color changes rapidly when we cross a place where one of the (X,Y,Z) coordinates is zero; these three loci are the intersections of the tilted sphere with coordinate planes. Appearances notwithstanding, this is indeed a *continuous* assignment of colors, and were it not for the discretization in RGB space, there would be a unique color assigned to each point in the complex plane. If we flatten the image to eight colors (my software says it contains more than 184 thousand), since each region corresponds to a vertex of the cube, we get a stereographic view of the polyhedron dual to the cube, namely the octahedron.

Using this color wheel in domain colorings gives a pleasantly art deco effect, as in Figure 5.

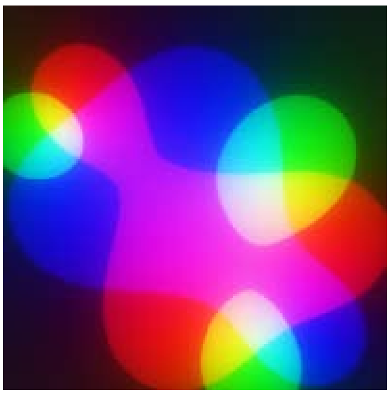

Figure 5: The cube-root color wheel used for a domain coloring of a cubic polynomial.

And now, the fifth dimension: The three-fold rotational symmetry of the cube-root color wheel is beautiful, and consistent with my original intention to place red, green, and blue around that unit circle at the cube roots of unity. Still, I wondered what other polyhedra might turn up as the result of more-or-less natural maps of the plane into the color cube. I wondered if I could produce the MAA logo, a regular icosahedron, in this manner. My first goal was to produce a color wheel with five-fold symmetry.

The most fruitful idea was to put the 2 x 2 x 2 cube into $\mathbb{R}^5$ so that it sits in a three-dimensional eigenspace of the cyclic permutation of five variables. I had observed in the past that this is a good source of five-fold symmetry. I used the components of stereographic projection as coefficients of normalizations of the vectors

$$\left(1,1,1,1,1\right),\left(1,\cos(2\pi/5),\cos(4\pi/5),\cos(6\pi/5),\cos(8\pi/5)\right),$$
$$\text{and}\left(0,\sin(2\pi/5),\sin(4\pi/5),\sin(6\pi/5).\sin(8\pi/5)\right).$$

Once the cube was placed in $\mathbb{R}^5$, I took cube roots of each component, moving each point toward the nearest vertex of the 5-cube. This pulls points off the 3-plane spanned by the three given vectors, so I simply projected them back. It was then easy to read off colors, in the same manner as before. The resulting color wheel is shown in Figure 6; actually, this one uses fifth roots instead of cube roots, which gives a more distinct coloration. Now that you know that a regular octahedron lurked in the previous color wheel, you might enjoy figuring out which polyhedron this represents.

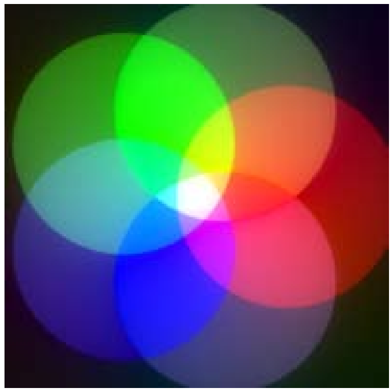

Figure 6: A color wheel with 5-fold symmetry, the source of the window design

There are 22 faces — two pentagons, ten triangles, and ten quadrilaterals. This is the dual of the 5-fold zonohedron. Given that there are 32 vertices of the 5-cube, why do points on the sphere inherit colors from only 22 of them? Computation shows that 10 of those vertices have small projection onto this 3-space; they are not sufficiently near to draw points.

I undid the stereographic projection and constructed this polyhedron in Maple, with the pentagonal faces missing so that you can look through. The faces are colored in only approximately the correct way. Figure 7 shows three views, from the top, the bottom, and along one of the great circles that corresponds to rings in Figure 6.

Figure 6 is the source of the window design, rotated for aesthetic effect. It is a window on the fifth dimension in this sense: Looking into the center, we are looking up into the portion of $\mathbb{R}^5$ where all coordinates are positive, along the vector (1,1,1,1,1).

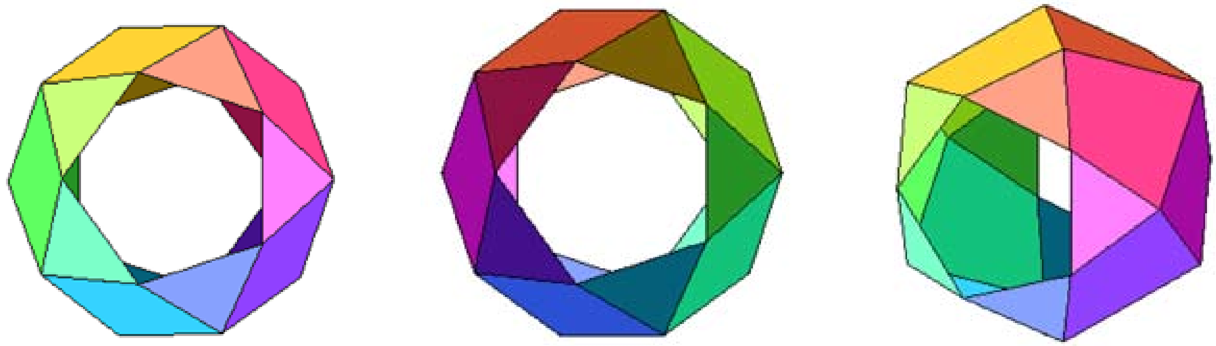

Figure 7: Three views of the dual of the 5-fold zonohedron

From there, crossing any of the five large rings takes us past a coordinate hyperplane, and one of the variables turns negative. When we have crossed all five rings, by any of the curious paths our eye can trace, we are looking back the other way, along (–1,–1,–1,–1,–1), toward darkness.

References

Frank A. Farris, Review of Tristam Needham's *Visual Complex Analysis* in *Amer. Math. Monthly*, 105:6 (1998), 570–576. (Supplementary materials, including color versions of the published images, appear at http://www.maa.org/pubs/amm_complements/complex.html.)

Frank A. Farris, Vibrating Wallpaper, *Communications in Visual Mathematics*, vol. 1 (later vol. 0 of *JOMA*).

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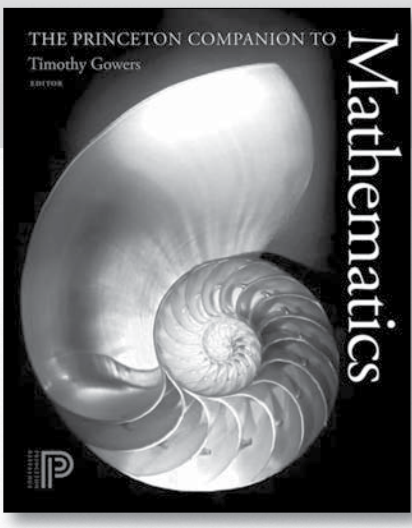

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